A PROBABILISTIC APPROACH TO PRIORITY ASSIGNMENT IN HIERARCHICALLY STRUCTURED DECISION PROBLEMS

By

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Samuel Y. Dennis, III

To my parents

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This dissertation considers the problem of assigning priorities over a set of activities under uncertainty. The research is based upon, and extends, the approach to modeling and deterministically analyzing complex decision problems developed by T.L. Saaty--the Analytic Hierarchy Process (AHP). The research extends the deterministic AHP paradigm to allow dominance judgments to vary probabilistically.

The case of a single criterion multi-activity decision problem is first considered. The joint probability distribution functions of the j^{th} column vector of the dominance matrix, and the principal right eigenvector are derived and their properties are presented and discussed in the context of the AHP paradigm. An elicitation procedure is advanced that enables the parameters characterizing the uncertain dominance judgments to be specified by the decision agent. The priority distribution of the various activities is derived, and its properties are presented and interpreted. A mathematical expression for the probability that the i^{th} activity of a set of n receives the highest

priority is developed. This expression provides a means for analyzing the robustness or stability of the highest priority ranking as the preference parameters characterizing the dominance judgments are allowed to vary through specified ranges.

The case of multi-criterion multi-activity decision problems is next considered. The probabilistic analysis of priority assignment in a simple hierarchically structured decision problem is conducted, and the limitations of the analysis are discussed.

Finally, the purpose of the dissertation and the underlying motivation for the development of the probabilistic extensions of the AHP paradigm are addressed. Specifically, the development of a decision analysis model of the Universities of Oklahoma and Georgia's decision to file suit against the National Collegiate Athletic Association. The objective is (1) to structure the decision problem of whether to file suit hierarchically in terms of general economic and sports related decision criteria and factors, and (2) to probabilistically analyze the assignment of priorities over the decision criteria and factors in order to determine the range of the importance preference judgments which impact upon the selection of the best activity or course of action.

CHAPTER I
INTRODUCTION

Introduction

In June 1984, the United States Supreme Court upheld the ruling of two lower federal courts that controls exercised by the National Collegiate Athletic Association (NCAA) over the broadcasting rights of its members as set forth in the "1982-85 NCAA Television Plan" violate the Sherman Antitrust Act, 15 U.S.C. §1-2 (1980). Consequently, contracts for live telecast of NCAA football games totaling some 280 million dollars over the 1982-85 playing seasons were invalidated. As a result, universities and colleges are free to negotiate autonomously for the assignment of television broadcasting rights for their football games. It is notable these significant events were triggered by the actions of only two of the 583 active members of the NCAA that field football teams, although sixty of the Division I NCAA membership actively sought to change the television control policies. Thus, the stimulus to this dissertation was the desire to gain insight into why so few decision agents in collegiate football participated in this momentous action.

The analysis of such a complex decision problem requires the use of a decision analysis methodology which (a) combines both the qualitative and quantitative aspects of the problem into a unified framework, and (b) is capable of generating a set of priorities for the various alternative courses of action. Such a methodology has been recently developed by T.L. Saaty (1977)--the Analytic Hierarchy Process (AHP). In the

course of developing a model capable of dealing with the various issues and perspectives addressed by this research, it became necessary to extend Saaty's deterministic AHP paradigm to allow the decision agent's judgments to vary probabilistically. Although the theoretical development of these extensions comprises the bulk of this research, the objective of this dissertation is twofold: (1) to structure the decision problem of whether to file suit hierarchically in terms of general economic and non-economic decision criteria and factors, and (2) to probabilistically analyze the assignment of priorities over the decision criteria and factors by varying the parameters characterizing their respective priority distributions. This investigative process is a highly flexible tool for systematically exploring the effects of a wide range of relative importance preference judgments upon the selection of the best activity or course of action. The purpose of the analysis is to learn more about the decision-making processes of collegiate athletic decision agents under uncertainty. The upshot is to make inferences about various combinations of priority assignments which substantiate the observed actions taken by the decision agents; the underlying goal being to gain some insight into why so few decision agents opted to take legal action against the NCAA.

Background

A frequently encountered problem of decision analysis is the assignment of weights or priorities over a set of activities. These weights reflect the complex notion of importance which must be defined carefully by specific decision-relevant criteria. Thus, the task of priority assignment is dependent upon the number and complexity of the defining criteria, as well as the number and type of activities under comparison. This task is further complicated if each criterion is not

shared by all of the activities, a situation that eliminates commonalities and makes inter-comparisons between activities more difficult.

The above complexities do not always arise in the practice of assigning priorities. For example, a decision agent may have a personal bias (either positive or negative) towards specific outcomes or activities; there may exist a single dominent or a clearly superior activity relative to others in the set; or the decision agent may be completely indifferent between the available activities. These "clear" decision problems preclude the necessity (rightly or wrongly) for use of decision analysis to reach a final decision. If these "clear" decision problems become delegated to others for solution, the terms obvious, superior, indifference, and preferred become problematic, and may require considerable care in definition and usage. Thus, a "clear" problem may require extensive analysis and thought by the naive decision maker charged with making a routine final decision. The virtual explosion in the demand for expert decision systems, and decision support systems in general, attests to the need for decision modeling and analysis of even the most routine delegated non-centralized decision problems.

The necessity for decision analysis is most apparent when problems of great complexity are encountered; that is, problems that challenge and tax the decision agent's logical capability to fully understand the causes and consequences of any action taken to solve them. These problems require systematic structuring and decomposition before the most elemental aspects of the problem can be comprehended and dealt with effectively. This is particularly true for multi-person, multi-criterion, and multi-period decision problems that involve many and different types of activities and varieties of outcomes. Decision problems require the use of judgment and experience on the part of decision agents for successful

structuring and analysis. Ideally, the analysis of such problems would combine both the quantitative and qualitative aspects of the problem into a unified framework capable of generating a set of priority for alternative courses of action.

A new approach to modeling and analyzing complex unstructured socio-economic decision problems has been developed by T.L. Saaty (1977)--the Analytic Hierarchy Process (AHP). The AHP uses quantitative and qualitative descriptions to hierarchically define a problem and to represent the interaction of its parts. The purpose of constructing hierarchies is to evaluate and prioritize the influence or impact of the various activities or alternatives on the attainment of a specified overall objective or goal. The AHP develops a framework for analyzing factors that affect the outcome of a complex decision problem and provides a measure of consistency of how well the inter-problem relationships and importances are understood by the decision agent.

Statement of Purpose

The AHP paradigm assumes that the expressed preferences of the decision agent are made under certainty. The purpose of this study is to extend the deterministic AHP paradigm to allow the decision agent's preference judgments to vary probabilistically; that is, to capture the decision agent's uncertainty regarding the relative importances of the various activities or alternatives under consideration. Rather than the assignment of precise numerical values for the priority weights, we have the assignment of random variables whose density functions represent the priority distributions of the constituent decision elements of the hierarchy. Also, instead of selecting the best activity based upon its numerical priority weight, the decision agent selects the best activity

based upon the properties characterizing the priority distributions of the activities under consideration.

Probabilistic analysis of the assignment of priorities permits decision problems to be "parameterized"; that is, the decision problem can be mathematically expressed in terms of the parameters characterizing the uncertain preference judgments of the decision agent. This feature enables the effects of changes in the parameter values to be evaluated in terms of the changes they induce in the priority rankings of the activities of interest. This form of sensitivity analysis is useful in establishing the robustness or stability of the ultimate ranking of the alternatives.

Probabilistic analysis of priority assignments also widens the scope of the type of decision problems that can be considered within the context of the AHP paradigm. The deterministic AHP paradigm restricts analysis to problems that directly involve the participation of all of the principal decision agents. Consequently, the paradigm is quite inappropriate if the purpose of a decision-making process is to draw inferences about the preference judgments of decision agents outside the circle of participation (i.e., competitors, policy-making bodies). Thus, the decision agent can become outward looking and investigative. For example, the investigative decision agent may posit probabilistic decision models that reflect perceptions of the critical problem elements facing the other decision agent under scrutiny or study. The posited decision models can be supplemented by (a) past observations or behavior of the agent under study, (b) public information, or (c) inferential information, either subjectively assessed through current observations, or obtained from independent experts, or consultants familiar with the decision-making process of the agent in question. Thus, the range of the possible values of the parameters characterizing the dominance judgments of the agent under

study can be evaluated in the light of the agent's choices of action.

Chapter Outlines

The next chapter presents the basic theory underlying the deterministic AHP paradigm for the case of cardinally transitive dominance ratios. The formal notions of a hierarchy and a priority function are discussed. The concept of a dominance matrix is introduced, and a method of scaling dominance ratios by using the principal right eigenvector of the dominance matrix is presented. The idea of a priority vector for a single criterion multi-activity decision problem is discussed, and its derivation from the principal right eigenvector of the dominance matrix is presented. A principle of hierarchical composition that extends the results of single criterion decision problems to hierarchically structured multi-criterion decision problems is also presented.

The third chapter extends the deterministic AHP paradigm discussed in Chapter II to allow the dominance judgments to vary probabilistically. The case of a single criterion multi-activity decision problem is considered. The joint probability distributions of the j^{th} column vector of the dominance matrix, and the principal right eigenvector are derived and their properties discussed. An elicitation procedure is advanced that enables the parameters characterizing the uncertain dominance judgments to be specified by the decision agent. The joint probability density of the priority vector (i.e., the priority distribution of the various activities) is derived and its properties are presented and interpreted.

The fourth chapter develops a mathematical expression for the probability that the i^{th} activity of a set of n receives the highest priority ranking, given the parameters characterizing the uncertain dominance judgments.

The fifth chapter extends the probabilistic AHP paradigm to include complex multi-criterion decision problems. The probabilistic analysis of priority assignment in a simple hierarchically structured decision problem is conducted, and the limitations of the analysis are presented and discussed.

The sixth chapter returns to the original motivating problem from collegiate athletics and addresses the aforementioned twofold objective of this dissertation.

CHAPTER II
AN INTRODUCTION TO THE ANALYTIC HIERARCHY PROCESS

Introduction

The Analytic Hierarchy Process[1] (AHP) is a theory of measurement and scaling that derives a vector of weights for a set of decision elements according to their relative importance or impact upon a given criterion, as evaluated by a decision agent. The decision elements may be various activities or subordinate criteria, and the criterion of interest may be some overall goal or objective that the decision elements are to achieve or meet. In complex unstructured multiple-criterion decision problems, the criteria themselves must be prioritized with respect to their importance relative to still higher sets of criteria. The nested arrangement of activities within criteria within supra-criteria, etc., defines a hierarchical structure consisting of various dependent levels of decision elements. The vector of weights derived from successive level-by-level prioritizations can be used as weighting factors for the priority weights of decision elements previously obtained at lower levels. In this way, the priority weights of decision elements on the various levels can be derived according to their impact upon the highest-level decision element of the hierarchy.

To further explain the process, consider a set of relevant decision elements and a given criterion. The fundamental problem is to

rank, according to their importance, each decision element relative to its impact upon the defining criterion. A matrix of pairwise comparisons can be constructed whose entries indicate the strength with which one decision element dominates another, relative to the criterion of interest to the evaluator. Scaling the ratios by using the principal right eigenvector of the positive pairwise comparison matrix yields the priority weight of each element with respect to the governing criterion. In multiple-decision problems, the decision elements in the lowest level of the hierarchy have a vector of weights with respect to each criterion in the next higher level. These criteria, however, have a vector of weights with respect to dominant criteria in the next higher level, and so on vertically through the hierarchical structure. The priority vectors at each level are combined to compose a priority matrix for that level. The priority matrix of each level is post-multiplied by the priority matrix of the next higher level. If the highest level of the hierarchical decision problem is a single element, then successive vertical priority matrix multiplications yield a single vector of weights. The entries of this vector indicate the relative priority of elements on the lowest level with respect to their impact upon the highest decision element of the hierarchy. If the decision problem at hand, for example, is to select a single activity that will achieve the highest objective, then the activity with the greatest priority weight can be chosen. If, however, scarce resources are to be allocated among several competing or dependent activities, then the priority vector indicates the proportion of the resources that needs to be allocated to each activity.

Hierarchies and Priority Functions

This section presents some basic definitions that serve to formally represent a hierarchy and a priority function. The fundamental decision problem of deriving priority weights of the lowest-level decision elements with respect to their impact upon the highest-level element of a hierarchy is discussed. A model for solving the fundamental problem is presented in the form of a principle of hierarchical composition.

Fundamental to the definition of a hierarchy is the notion of a partial order. A partial order in a set X is a reflexive, antisymmetric and transitive relation in X. A partial order in a set X may be defined as a relation $\leq$ in X such that, for all x, y, and z in X, (i) $x \leq x$, (ii) if $x \leq y$ and $y \leq x$, then $x = y$, and (iii) if $x \leq y$ and $y \leq z$, then $x \leq z$. A partially ordered set is an ordered pair $\langle X, \leq \rangle$, such that $\leq$ partially orders X (see Halmos, 1960, p. 54).

Suppose that $\leq$ partially orders X, and that the elements x and y are contained in X. If $x \leq y$ and $x \neq y$, then we write $x < y$, and we may say that x is less than y (y is greater than x) or x is a predecessor of y (y is a successor of x). If $x < y$ and there is no element u in X such that $x < u < y$, we say that x is an immediate predecessor of y (y is an immediate successor of x), or that y is a cover of x in X. The set $S(y) = \{x \in X: \exists \sim u \in X \ni x < y < u\}$ is the set of immediate predecessors of element y in X.

If X is a partially ordered set, and if $b \in X$, the set $\overline{S}(b) = \{x \in X: x < b\}$ is termed the initial segment determined by b. If the element a in X is such that $x \leq a$ for all x in X, then a is called

the greatest (largest, last) element of X. The antisymmetry property of the partial order relation implies that if X has a greatest element, then it has only one such element (see Halmos, 1960, p. 56). The set $\overline{S}(a) = \{x \in X: x < a\}$ is the set of predecessors of the greatest element a (see Halmos, 1960, p. 56). It follows that $\overline{S}(a) = S(a) \cup S^2(a) \cup S^3(a) \cup \ldots$, where $S^2(a) = S(S(a))$, $S^3(a) = S(S(S(a)))$, etc. The set S(a) represents the set of immediate predecessors of a, and the set S(S(a)) represents the set of immediate predecessors of the immediate predecessors of a, etc. The sets $S^i(a)$, i = 1,...,n, are disjoint sets whose union collectively exhausts $\overline{S}(a) \subseteq X$.

Let H be a finite partially ordered set. If H has a greatest element a, then H is called a hierarchy. That is, H is a disjoint collection of non-empty subsets (called levels) $L_i = S^i(a)$, i = 1,...,n, and $L_0 = \{a\}$. Thus, the set $H = \bigcup_{i \in I} L^i(a)$, where the set $I = \{0,1,\ldots\}$ indexes the levels of H.

Assume there exists a function $w: H \times H \to [0,1]$, such that:

$$\text{(i)} \quad w(x,y) = w_x(y) = \begin{cases} 0 \ , & y \notin \overline{S}(x) \qquad (2.1a) \\ \geq 0 \ , & y \in \overline{S}(x) \qquad (2.1b) \end{cases}$$

$$\text{(ii)} \quad \sum_{y \in S(x)} w_x(y) = 1 \qquad (2.2)$$

and

$$\text{(iii)} \quad w_x(z) = \sum_{y \in S(x)} w_x(y) w_y(z), \quad x \in S^2(x). \qquad (2.3)$$

The function $w_x(y)$ is called the weighting function or the priority function of the elements y in the initial segment determined by x.

Thus, the priority of any element $z \in S^2(x)$ with respect to $x \in L_i$ is found by weighting the priority of element z relative to the immediate successors $y \in S(x)$ by the weight of each $y \in S(x)$ upon the objective x.

Suppose, for illustrative purposes, that for each $x \in L_i$ we have $S(x) = \{y_1,\ldots,y_m\}$ and $S^2(x) = \{z_1,\ldots,z_p\}$. Equation (2.3) can be written as

$$w_x(z_k) = \sum_{j=1}^{m} w_x(y_j) w_{y_j}(z_k) \qquad k = 1,\ldots,p. \tag{2.4}$$

Inspection of equation (2.4) reveals that it may be rewritten as

$$W = BQ \tag{2.5}$$

where

$$B = \{w_{y_j}\} = \{w_{y_j}(z_k)\} \tag{2.6a}$$

$$Q = \{w_x(y_j)\} \tag{2.6b}$$

$$W = \{w_x(z_k)\} \tag{2.6c}$$

for k = 1,...,p, and j = 1,...,m. The (p × m) matrix B is called the priority matrix of the $(i + 2)^{th}$ level. The components of the column vectors of B (i.e., $w_{y_j}(z_k)$, j = 1,...,m) represent the priority of each $z \in S^2(x)$ with respect to each of its immediate successors $y \in S(x)$. The (m × 1) column vector Q is called the priority vector of the $(i + 1)^{th}$ level. The components of this vector represent the priority of each of the immediate predecessors of x. Consequently, the (p × 1) column vector W represents the priority vector of the elements $z \in S^2(x)$ with respect to the objective x.

The above priority weighting scheme is referred to as hierarchical composition by Saaty (see Saaty, 1977, pp. 263-64). The process can be extended to find the priority of any element $z \in \overline{S}(x)$. For example, suppose that $z = x_m < x_{m-1} < \ldots < x_{i+1} < x_i = x$, where $x_j \in \overline{S}(x)$, and x_j covers x_{j+1} for $j = i+1,\ldots,m$. The priority function $w_x(z)$ can be expressed as

$$w_{x_i}(x_m) = \sum_{x_{i+1},\ldots,x_{m-1} \in \overline{S}(x_i)} \ldots \sum w_{x_i}(x_{i+1}) \ldots w_{x_{m-1}}(x_m), \quad x_m \in S^{m-1}(x_i). \tag{2.7}$$

Equation (2.7) can be written in vector form as

$$W = B_m B_{m-1} \ldots B_{i+1} Q. \tag{2.8}$$

If $x = L_0 = \{a\}$, and $m = n$, then the priority vector of the lowest level predecessors of the element a can be written as

$$W = B_n B_{n-1} \ldots B_1 Q. \tag{2.9}$$

Principal Right Eigenvector Scaling of Dominance Ratios of a Consistent Pairwise Comparison Matrix

This section presents the method of scaling dominance ratios by using the principal right eigenvector of a judgmentally consistent

pairwise comparison matrix. That is, the entries of the pairwise comparison matrix are assumed to satisfy the property of cardinal transitivity. The general case of judgmentally intransitive but reciprocal responses is not treated in this research.

Consider the fundamental problem of ranking a set of decision elements $S = \{s_1,\ldots,s_n\}$ with respect to their influence on some objective or criterion. A matrix of pairwise comparisons $A = \{a_{ij}\}_{i,j=1,\ldots,n}$ can be constructed whose entries a_{ij}, $i,j = 1,\ldots,n$, indicate the strength with which decision element s_i dominates element s_j with respect to their influence upon the given criterion. In practice, a decision maker would assign numbers to each pairwise comparison using a numerical scale that reflects the decision maker's perception of the magnitude of difference in importance among the various decision elements. Saaty (1977) has suggested an intensity scale whose values correspond to various importance ranks; this scale is shown in Table 2.1.

If the pairwise comparison judgments are consistent, then the entries of the matrix A satisfy the property of cardinal transitivity, that is,

$$a_{ik} = a_{ij}a_{jk}, \qquad i,j,k = 1,\ldots,n. \tag{2.10}$$

Thus, cardinal transitivity suggests that each pairwise comparison a_{ij} can be regarded as a ratio of the underlying "absolute" weights of importance of the decision elements s_i and s_j, respectively. That is,

$$a_{ij} = w_i/w_j, \qquad i,j = 1,\ldots,n. \tag{2.11}$$

This suggests that the matrix $A = \{a_{ij}\}_{i,j=1,\ldots,n}$ has the form

Table 2.1
The Saaty Response Scale for Dominance
Paired Comparison Judgments

Intensity of importance	Definition	Explanation
1	Equal importance	Two activities contribute equally to the objective
3	Weak importance of one over another	Experience and judgment slightly favor one activity over another
5	Essential or strong importance	Experience and judgment strongly favor one activity over another
7	Demonstrated importance	An activity is strongly favored and its dominance is demonstrated in practice
9	Absolute importance	The evidence favoring one activity over another is of the highest possible order of affirmation
2,4,6,8	Intermediate values between the two adjacent judgments	When compromise is needed
Reciprocals of above nonzero	If activity i has one of the above nonzero numbers assigned to it when compared with activity j; then j has the reciprocal value when compared with i	
Rationals	Ratios arising from the scale	If consistency were to be forced by obtaining n numerical values to span the matrix

$$A = \begin{bmatrix} w_1/w_1 & w_1/w_2 & \cdots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \cdots & w_2/w_n \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ w_n/w_1 & w_n/w_2 & \cdots & w_n/w_n \end{bmatrix} = \left\{\frac{w_i}{w_j}\right\}_{i,j=1,\ldots,n}. \qquad (2.12)$$

More to the point, A can be written as

$$A = w(w^{-1})^t = \begin{pmatrix} w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_n \end{pmatrix} \left(\frac{1}{w_1},\ldots,\frac{1}{w_j},\ldots,\frac{1}{w_n}\right). \qquad (2.13)$$

Since any column of A is a constant (positive) multiple of the vector w, A must have unit rank.

The method of scaling the dominance ratios by using the principal right eigenvector of the pairwise comparison matrix requires searching for positive scalars $\lambda(z)$ for which there exist vectors $z \neq 0$ satisfying the equation

$$Az = \lambda(z)z. \qquad (2.14)$$

Since $A = w(w^{-1})^t$, we can rewrite the latter equation as

$$Az = w(w^{-1})^t z = \lambda(z)z. \qquad (2.15)$$

Because of their dimensionality, the vector product of $(w^{-1})^t$ and z is a nonzero scalar, call it β. Then the above equation becomes

$$Az = \beta w = \lambda(z)z. \qquad (2.16)$$

Thus, the vector w is an eigenvector of the matrix A. To show that w is the principal right eigenvector of A, let A be a matrix representation of a linear operator over a vector space.

Let V be a vector space over the scalar field F, and let T be a linear operator on V, that is, $T: V \to V$. A nonzero vector $z \in V$ is called an eigenvector of T if there exist scalars $\lambda(z) \in F$ such that

$$Tz = \lambda(z)z. \tag{2.17}$$

The range, null space, rank, and nullity of the linear operator T are defined, respectively, as

$$\text{range } (T) = \{Tz: z \in V\}, \tag{2.18}$$

$$\text{null space } (T) = \{z \in V: Tz = 0\}, \tag{2.19}$$

$$\text{rank } (T) = \dim [\text{range } (T)], \tag{2.20}$$

$$\text{nullity } (T) = \dim [\text{null space } (T)], \tag{2.21}$$

where dim (dimension) of a vector space W is the number of linearly independent vectors spanning W. It is well known (see Halmos, 1974, p. 90) that

$$\text{rank } (T) + \text{nullity } (T) = \dim (V). \tag{2.22}$$

But dim (V) = n, since the size of the ordered basis spanning the vector space V is n. Also, rank (T) = 1, since the rank of a linear transformation T is equal to the rank of the matrix representing T; that is, rank (T) = rank (A) = 1. This implies nullity (T) = n - 1. Thus,

the range of T is spanned by a single nonzero eigenvector z whose eigenvalue $\lambda(z)$ is nonzero. In fact, all nonzero vectors in the kernel of T are also eigenvectors; however, z is the only eigenvector having a nonzero eigenvalue (z is referred to as the principal right eigenvector). From a previous discussion is was found that $z = w$, thus implying that w is the principal right eigenvector of A.

Thus, the underlying vector of the absolute weights w of the decision elements can be recovered by solving the system of equations $Az = \lambda(z)z$ in the unknown z. For comparison purposes it is convenient to normalize the components of w such that they sum to unity. The resulting normalized principal right eigenvector is referred to as the priority vector. The elements of the priority vector represent the relative strength or priority of each decision element in relation to their impact upon the criterion of interest. Fortunately, since A is a consistent matrix any column of A solves $Az = \lambda(z)$. Thus, in this situation any normalized column of A yields the priority vector directly.

Note

[1] See Saaty (1977), pp. 234-281.

CHAPTER III
A PROBABILISTIC MODEL FOR EIGENVECTOR SCALING
OF CONSISTENT DOMINANCE RATIOS

Introduction

The purpose of this chapter is to extend the deterministic AHP paradigm to allow the pairwise comparison judgments to vary probabilistically. Consequently, the dominance ratio entries of the pairwise comparison matrix are treated as random variables that capture the decision agent's uncertainty regarding the relative importances of the various decision elements under consideration. Uncertainty may arise for a variety of reasons. For instance, a vaguely enunciated decision problem may confound attempts to quantify relative importance judgments by even the most expert of decision agents. This situation is further exacerbated by the plethora of vague complex problems that confound enunciation in the first place. In addition, an adequately structured decision problem does not ensure that decision agents will not find it difficult to meticulously articulate perceived degrees of relative importance among the various decision elements under comparison. More interestingly, consider the case of a decision agent (who may well be ourselves) who wishes to learn more about how other decision agents (e.g., competitors or policy-making bodies) make decisions. In this situation, the investigative agent must posit surrogate decision models

that act to simulate the decision models used by the other agents (if indeed any such models are used, either directly or indirectly, by the other agents at all). Even if the appropriateness of such posited models is not in doubt, the investigative agent must, perforce, make uncertain inferences about possibly (and most likely) uncertain importance judgments of the other agents--a fact that further complicates analysis. It is these types of situations that engender the need to allow the dominance ratios to vary probabilistically to attempt to capture the problem elements of uncertainty and ambiguity. This in no way means that these kinds of problems can be satisfactorily answered, but it is hoped that they can at least be treated soundly at an elementary level. The present probabilistic extensions consider the case of a consistent random dominance matrix. The general case of reciprocal dominance matrices is not considered, due to the severe analytical difficulties encountered in the distributional analysis of the eigenvector problem for reciprocal matrices of arbitrary rank. At this point, however, it is convenient to present a brief summary of the basic properties of a random consistent pairwise comparison matrix, and to discuss various criteria that influence the choice of a family of distributions to represent the principal right eigenvector random variables.

Let $\tilde{X} = \{\tilde{x}_{ij}\}_{i,j=1,\ldots,n}$ be a matrix of pairwise comparisons whose entries are positive random variables. If $\tilde{X}$ is a consistent matrix, it can be written as

$$\tilde{X} = (\tilde{v})(\tilde{v}^{-1})^t = \{\tilde{v}_i/\tilde{v}_j\}_{i,j=1,\ldots,n} \tag{3.1}$$

where the vector random variable $\tilde{v} = (\tilde{v}_1,\ldots,\tilde{v}_n)^t$ is the principal right eigenvector of $\tilde{X}$, and $\tilde{v}^{-1} = (1/\tilde{v}_1, 1/\tilde{v}_2,\ldots,1/\tilde{v}_n)^t$.

The random variables $\tilde{v}_i$, $i = 1,\ldots,n$, represent the decision agent's uncertain perceptions of the absolute importances of the various decision elements under consideration. Hence, the $\tilde{v}_i$ can be considered to be independent positive random variables. Consequently, the entries of the matrix $\tilde{X}$ are quotients of independent positive random variables that capture the underlying process of making relative importance judgments. In more formal terms, the random variable entries $\tilde{x}_{ij}$, $i,j = 1,\ldots,n$, of a consistent dominance matrix $\tilde{X}$ must satisfy the following properties:

(i) $\tilde{x}_{ij} = \tilde{v}_i/\tilde{v}_j$;

(ii) $\tilde{x}_{ik} = \tilde{x}_{ij}\tilde{x}_{jk}$, implying that $\tilde{x}_{ij} = (\tilde{x}_{ji})^{-1}$ for all $i,j,k = 1,\ldots,n$, where the v_i are independent positive random variables.

The choice of a family of distribution functions to represent the principal right eigenvector random variables $\tilde{v}_i$ specifies the derived distribution functions of the random variable entries of the matrix $\tilde{X}$. Consequently, the choice of the former distribution should be conditioned upon meeting the following qualitative "model" criteria: (1) the derived distribution function for the random variables representing the dominance ratios $\tilde{x}_{ij}$ should adequately model some reasonable representation of a relative importance intensity response scale; and (2) the

derived distribution function of the normalized principal right eigenvector should lead to subsequent tractable distributional analysis, as well as contribute to interpretational flexibility and model richness.

The above criteria are sufficiently satisfied if the $\tilde{v}_i$ are assumed to be distributed as gamma variates having parameter α_i, common parameter β (denoted as $\tilde{v}_i \sim \Gamma(\alpha_i,\beta)$), and probability density function (p.d.f.)

$$f_i(v_i) = \begin{cases} \dfrac{v_i^{\alpha_i-1} e^{-v_i/\beta}}{\beta^{\alpha_i}\Gamma(\alpha_i)}, & v_i > 0 \quad \begin{bmatrix} \alpha_i > 0 \\ \beta > 0 \end{bmatrix} \\ 0, & \text{elsewhere} \end{cases} \tag{3.2}$$

for $i = 1,\ldots,n$. As a result, the quotients

$$\tilde{x}_{ij} = \tilde{v}_i/\tilde{v}_j \qquad i = 1,\ldots,n,\ i \neq j \tag{3.3}$$

are distributed as inverted beta-2 variates having parameters α_i and α_j (denoted as $\tilde{x}_{ij} \sim i\beta_2(\alpha_i,\alpha_j)$) with the p.d.f.

$$g(x_{ij}) = \begin{cases} \dfrac{\Gamma(\alpha_i+\alpha_j)}{\Gamma(\alpha_i)\Gamma(\alpha_j)} \dfrac{x_{ij}^{\alpha_i-1}}{(1+x_{ij})^{\alpha_i+\alpha_j}}, & x_{ij} > 0 \\ 0, & \text{elsewhere} \end{cases} \tag{3.4}$$

for $i,j = 1,\ldots,n,\ i \neq j$.

In order to examine the distributional consequences of the assumption that $\tilde{v}_i$ is gamma-distributed, the joint probability density function (j.p.d.f.) and the marginal probability density functions (m.p.d.f.'s)

associated with the j^{th} column vector ($j = 1,\ldots,n$) of the dominance matrix $\tilde{X}$, and the normalized principal right eigenvector are developed and their properties presented. In addition, the derived distributional results are interpreted in the context of the AHP paradigm. The mathematical appendix provides the theoretical arguments that establish distributional characterizations.

The Distribution of the j^{th} Column Vector of the Pairwise Comparison Matrix

This section is concerned with formally deriving the j.p.d.f. of the j^{th} column vector of the dominance matrix $\tilde{X}$ from the principal right eigenvector. The m.p.d.f. of the i^{th} element of the j^{th} column (that is, $\tilde{x}_{ij}$, $i,j = 1,\ldots,n$, $i \neq j$) is also derived. Distributional properties of these density functions are also presented and interpreted. A method of determining the distributional parameters characterizing the m.p.d.f.'s of the $\tilde{x}_{ij}$'s from a series of decision agent supplied uncertain relative importance judgments is also discussed. The existence of synonymous families of distribution functions whose random variable quotients yield the same m.p.d.f. for the $\tilde{x}_{ij}$'s is examined, as are the necessary and sufficient conditions that uniquely characterize the distribution of the principal right eigenvector random variables from the joint distribution of any of the column vectors of the dominance matrix $\tilde{X}$.

Suppose the $\tilde{v}_i$ are independently distributed gamma variates with density (3.2). Then, the j.p.d.f. of the vector random variable

$\tilde{v} = (\tilde{v}_1,\ldots,\tilde{v}_n)$ is uniquely determined by the product of the m.p.d.f.'s, that is,

$$f_v(v_1,\ldots,v_n) = \prod_{i=1}^{n} f_i(v_i) = \prod_{i=1}^{n} \frac{v_i^{\alpha_i-1}}{\beta^{\alpha_i}\Gamma(\alpha_i)} \left(\exp\left(-\frac{1}{\beta}\sum_{j=1}^{n} v_j\right)\right) \qquad (3.5)$$

for $v_i > 0$ $(\alpha_i, \beta > 0)$, $i = 1,\ldots,n$.

If $\tilde{X}$ is a judgmentally consistent dominance matrix, then it has rank one and can be written as the product of two vector random variables as in equation (3.1). Thus, the j^{th} column of the matrix $\tilde{X}$, denoted by the vector random variable $\tilde{x}_j = \{\tilde{x}_{1j},\ldots,\tilde{x}_{(j-1)j},\tilde{x}_{(j+1)j},\ldots,\tilde{x}_{nj}\}$, $j = 1,\ldots,n$, can be obtained from equation (3.5) by applying the transformation given by

$$T: \tilde{x}_{ij} = \tilde{v}_i/\tilde{v}_j \qquad (i,j = 1,\ldots,n,\ i \neq j) \qquad (3.6)$$

so that

$$T^{-1}: \tilde{v}_i = \tilde{v}_j\tilde{x}_{ij} \qquad (i,j = 1,\ldots,n,\ i \neq j). \qquad (3.7)$$

For notational simplification and ease of analysis, it is convenient to consider the case of $j = n$, where (3.6) and (3.7) can be rewritten, respectively, as follows:

$$T: \tilde{x}_{in} = \tilde{v}_i/\tilde{v}_n \qquad (i = 1,\ldots,n-1) \qquad (3.8)$$

and

$$T^{-1}: \tilde{v}_i = \tilde{v}_n\tilde{x}_{in} \qquad (i = 1,\ldots,n-1) \qquad (3.9)$$

$$\sum_{i=1}^{n-1} \tilde{v}_i = \tilde{v}_n \ (1 + \sum_{i=1}^{n-1} \tilde{x}_{in}).$$

The Jacobian of this transformation is

$$J = \frac{\partial(\tilde{v}_1,\ldots,\tilde{v}_{n-1})}{\partial(\tilde{x}_{1n},\ldots,\tilde{x}_{(n-1)n})}$$

$$= \begin{vmatrix} \tilde{v}_n & 0 & \ldots & 0 & 0 \\ 0 & \tilde{v}_n & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & \tilde{v}_n & 0 \\ 0 & 0 & \ldots & 0 & \tilde{v}_n \end{vmatrix}$$

$$= \tilde{v}_n^{n-1}. \tag{3.10}$$

Application of the transformation (3.8) (that is, substituting equations (3.9) into (3.5) and multiplying the result by (3.10)) yields, upon rearranging, the j.p.d.f. of the vector random variable $(\tilde{x}_{1n},\ldots,\tilde{x}_{(n-1)n},\tilde{v}_n)$,

$$h(x_{1n},\ldots,x_{(n-1)n},v_n) = \frac{\prod_{i=1}^{n-1} (x_{in})^{\alpha_i - 1}}{\prod_{i=1}^{n} \Gamma(\alpha_j)\beta^{\alpha_j}} \ v_n^{\sum_{j=1}^{n} \alpha_j - 1} \exp\left[-\frac{v_n}{\beta}(1 + \sum_{i=1}^{n-1} x_{in})\right]$$

$$(x_{in} > 0,\ i = 1,\ldots,n-1;\ v_n > 0)\ . \tag{3.11}$$

Integrating equation (3.11) with respect to $\tilde{v}_n$ (recognizing that $\tilde{v}_n \sim \Gamma(\sum_{j=1}^{n} \alpha_j, \beta/(1 + \sum_{i=1}^{n-1} \tilde{x}_{in}))$ and rearranging yields the j.p.d.f. of the n^{th} column vector of the pairwise comparison matrix $\tilde{X}$; that is

$$h_{x_n}(x_{1n},\ldots,x_{(n-1)n}) = \frac{\Gamma(\sum_{i=1}^{n} \alpha_i)}{\prod_{i=1}^{n} \Gamma(\alpha_i)} \prod_{j=1}^{n-1} x_{jn}^{\alpha_j - 1} \left(1 + \sum_{i=1}^{n-1} x_{in}\right)^{-\sum_{i=1}^{n} \alpha_i} \tag{3.12}$$

for $0 < x_{in} < \infty$, $i = 1,\ldots,n-1$, in R_{n-1} and zero elsewhere. Tiao and Guttman (1965, p. 794) refer to a distribution having j.p.d.f. (3.12) as the (n-1)-variate inverted Dirichlet distribution $D'(\alpha_1,\ldots,\alpha_{n-1};\ \alpha_n)$.

The general moment $\mu'_{r_1\ldots r_{n-1}}$ of the (n-1)-variate inverted Dirichlet distribution has the value

$$\mu'_{r_1\ldots r_{n-1}} = \frac{\Gamma(\alpha_n - \sum_{j=1}^{n-1} r_j)}{\Gamma(\alpha_n)} \prod_{j=1}^{n-1} \frac{\Gamma(\alpha_j + r_j)}{\Gamma(\alpha_j)} \tag{3.13}$$

if

$$\alpha_n > \sum_{i=1}^{n-1} r_j, \tag{3.14}$$

and does not exist elsewhere. In particular the means, variances, and the covariances of the $\tilde{x}_{in}$'s are

$$E(\tilde{x}_{in}) = \frac{\alpha_i}{\alpha_n - 1}, \quad (\alpha_n > 1), \quad i = 1,\ldots,n-1 \tag{3.15}$$

$$V(\tilde{x}_{in}) = \frac{\alpha_i(\alpha_i + \alpha_n - 1)}{(\alpha_n - 1)^2(\alpha_n - 2)}, \quad (\alpha_n > 2), \quad i = 1,\ldots,n-1 \tag{3.16}$$

$$\mathrm{Cov}(\tilde{x}_{in},\tilde{x}_{jn}) = \frac{\alpha_i\alpha_j}{(\alpha_n - 1)^2(\alpha_n - 2)}, \quad (\alpha_n > 2), \quad i \neq j = 1,\ldots,n-1. \tag{3.17}$$

An important property of the (n-1)-variate inverted Dirichlet distribution is[1]

Property 3.1. If $(\tilde{x}_{1n},\ldots,\tilde{x}_{(n-1)n})$ is a vector random variable having the (n-1)-variate inverted Dirichlet distribution $D'(\alpha_1,\ldots,\alpha_{n-1};\ \alpha_n)$, then the marginal distribution of $(\tilde{x}_{1n},\ldots,\tilde{x}_{kn})$, $k < n-1$, is the k-variate inverted Dirichlet distribution $D'(\alpha_1,\ldots,\alpha_k;\ \alpha_n)$.

It is apparent from this property that the $\tilde{x}_{ij}$, $i \neq j$, are distributed as inverted beta-2 variates having parameters α_i and α_j with density function (3.4). The mean and variances of the $\tilde{x}_{ij}$'s are given in equations (3.15) and (3.16), respectively.

The random variables $\tilde{x}_{ij}$ represent the decision agent's uncertain judgments regarding the relative importance of element $i = 1,\ldots,n$, $i \neq j$, compared to element j, according to their impact upon some criterion of concern. The parameters α_i, α_j characterizing the density function (3.4) can be determined from a series of judgments made by the decision agent; specifically, the specification of perceived values of both the expected value and variance of the inverted beta-2 distribution whose variate represents the uncertain pairwise comparison judgment. Using this information equations (3.15) and (3.16) can be solved to determine the values of the unknown parameters α_i, α_j. For illustrative purposes, assume that $\alpha_j >> 2$, $E(\tilde{x}_{ij}) = n'$ and $V(\tilde{x}_{ij}) = (\sigma')^2$, where n' and $(\sigma')^2$ are supplied by the decision agent.

Equations (3.15) and (3.16) can then be rewritten, respectively, as

$$E(\tilde{x}_{ij}) = \frac{\alpha_i}{\alpha_j} = n' \tag{3.18}$$

$$V(\tilde{x}_{ij}) = \frac{n'(n' + 1)}{\alpha_j} = (\sigma')^2. \tag{3.19}$$

These equations can be readily solved for α_i and α_j, subject to $\alpha_j >> 2$. It can be seen that the capability to manipulate both parameters of the inverted beta-2 density allows for great flexibility in modeling uncertain relative dominance judgments, and hence satisfies the first of the "model" criteria enunciated earlier in this chapter.

It is well known that if two independent random variables $\tilde{v}_i$, $\tilde{v}_j$ are gamma distributed with density (3.2), then their quotients (3.3) are distributed as inverted beta-2 variates having density (3.4). This property, however, does not characterize the gamma distribution uniquely. Kotlarski (1962) has proved that there exist pairs of independent positively distributed random variables $\tilde{v}_i$, $\tilde{v}_j$ whose common distribution function $1(v)$ differs from the one given by density (3.2) but whose quotients (3.3) are distributed according to density (3.4). Some such distribution functions are given by the following densities:

$$1_1(v) = \begin{cases} \dfrac{1}{\beta^{\alpha}\Gamma(\alpha)} \dfrac{e^{-1/\beta v}}{v^{(\alpha + 1)}}, & \text{for } v > 0 \\ 0, & \text{elsewhere} \end{cases} \tag{3.20}$$

$$l_2(v) = \begin{cases} \dfrac{2\Gamma(\alpha + 1/2)}{\Gamma(\alpha/2)\Gamma(\alpha + 1)/2} \dfrac{v^{\alpha-1}}{(1 + v^2)^{\alpha + 1/2}}, & \text{for } v > 0 \\ 0, & \text{elsewhere} \end{cases} \tag{3.21}$$

and

$$l_3(v) = \begin{cases} \dfrac{2\Gamma(\alpha + 1/2)}{\Gamma(\alpha/2)\Gamma(\alpha + 1)/2} \dfrac{v^{\alpha}}{(1 + v^2)^{\alpha + 1/2}}, & \text{for } v > 0 \\ 0, & \text{elsewhere.} \end{cases} \tag{3.22}$$

Kotlarski (1967) proved that the j.p.d.f. of the quotients

$$\tilde{x}_{ik} = \tilde{v}_i/\tilde{v}_k, \ \tilde{x}_{jk} = \tilde{v}_j/\tilde{v}_k \tag{3.23}$$

(where the random variables $\tilde{v}_i$, $\tilde{v}_j$, and $\tilde{v}_k$ are independent gamma variates with density (3.2)) characterizes (by some assumptions) the density function (3.2) uniquely, up to a change in scale. This characterization of the gamma distribution is stated in the following theorem:

Theorem 3.1 (Kotlarski, 1967). Let $\tilde{v}_1,\ldots,\tilde{v}_n$ be positive independent random variables, and let $\tilde{x}_{ij} = \tilde{v}_i/\tilde{v}_j$ for $i,j = 1,\ldots,n$, $i \neq j$. The necessary and sufficient conditions for $\tilde{v}_k$, $k = 1,\ldots,n$ to be gamma distributed variates with parameters α_i and β (β-common, $k = 1,\ldots,n$) is that the vector random variable $\tilde{x}_j = (\tilde{x}_{ij},\ldots,\tilde{x}_{(j-1)j}, \tilde{x}_{(j+1)j},\ldots,\tilde{x}_{nj})$ be distributed as the $(n-1)$-variate inverted Dirichlet distribution $D'(\alpha_1,\ldots,\alpha_{j-1},\alpha_{j+1},\ldots,\alpha_n; \alpha_j)$ with density

$$h_{x_j}(x_{ij},\ldots,x_{(j-1)j},x_{(j+1)j},\ldots,x_{nj}) =$$

$$\frac{\Gamma(\sum_{i=1}^{n} \alpha_j)}{\prod_{i=1}^{n} (\Gamma(\alpha_i))} \prod_{\substack{i=1 \\ i\neq j}}^{n} x_{ij}^{\alpha_i - 1} \left(1 + \sum_{\substack{i=1 \\ i\neq j}}^{n} x_{ij}\right)^{-\sum_{i=1}^{n} \alpha_i} \tag{3.24}$$

for $\tilde{x}_{ij} > 0$, i,j = 1,...,n, i ≠ j, and zero elsewhere.

This theorem is of great theoretical significance to the probabilistic extension of the deterministic AHP paradigm considered in this research. Specifically, the joint distribution function of the normalized principal right eigenvector is not uniquely specified by the marginal distribution functions of the random variable entries of the pairwise comparison matrix $\tilde{X}$. To uniquely specify this joint distribution function it is necessary to specify the joint distribution function of any one of the column vectors of the dominance matrix $\tilde{X}$. The latter specification also uniquely specifies the distribution of the principal right eigenvector random variables.

The Distribution of the Normalized Principal Right Eigenvector

This section is concerned with the distributional properties of both the j.p.d.f. and the m.p.d.f.'s of the normalized principal right eigenvector as derived from the principal right eigenvector, and from the j^{th} column vector of the pairwise domparison matrix $\tilde{X}$. As mentioned in Chapter II, the normalized principal right eigenvector represents the relative strength or priority of each decision element under comparison. Analogously, under uncertainty, the distribution of the principal right eigenvector can be thought of as representing the priority distribution of each decision element according to its impact or influence on the governing criterion. The parameters of the priority distributions are determined via the characterization of the elements of the pairwise

comparison matrix $\tilde{X}$ (as mentioned previously in this chapter). The distributional properties of the normalized principal right eigenvector aid in interpreting and understanding the effects of uncertainty on the priority rankings of the relevant decision elements, as well as provide a means of analyzing the sensitivity of such rankings to variations in the parameters characterizing the pairwise comparison judgments. In addition to discussing the various distributional properties of the priority vector, a clustering or aggregation theorem that has important implications for probabilistic modeling of complex multi-criterion, multi-element decision problems is also presented.

If the $\tilde{v}_i$ are distributed as independent gamma variates having density (3.2), then the j.p.d.f. of the vector random variable $\tilde{v} = (\tilde{v}_1,\ldots,\tilde{v}_n)$ is given by density

$$f_v(v_1,\ldots,v_n) = \prod_{i=1}^{n} f_i(v_i) = \prod_{i=1}^{n} \frac{v_i^{\alpha_i-1}}{\beta^{\alpha_i}\Gamma(\alpha_i)} (\exp(-\frac{1}{\beta}\sum_{j=1}^{n} v_j)) \qquad (3.25)$$

for $v_i > 0$ $(\alpha_i,\beta > 0)$, $i = 1,\ldots,n$.

The j.p.d.f. of the normalized principal right eigenvector can be obtained from density (3.25) by applying the transformation

$$T: \tilde{y}_i = \frac{\tilde{v}_i}{\beta\tilde{y}_n}, \quad i = 1,\ldots,n-1; \quad \tilde{y}_n = \frac{1}{\beta}\sum_{i=1}^{n} \tilde{v}_i \qquad (3.26)$$

so that

$$T^{-1}: \frac{\tilde{v}_i}{\beta} = \tilde{y}_i\tilde{y}_n, \quad i = 1,\ldots,n-1; \quad \frac{\tilde{v}_n}{\beta} = \tilde{y}_n(1 - \sum_{i=1}^{n-1} \tilde{y}_i). \qquad (3.27)$$

Tha Jacobian of this transformation is

$$J = \frac{\partial(\tilde{v}_1/\beta,\ldots,\tilde{v}_n/\beta)}{\partial(\tilde{y}_1,\ldots,\tilde{y}_n)}$$

$$= \begin{vmatrix} \tilde{y}_n & 0 & 0 & \cdots & 0 & \tilde{y}_1 \\ 0 & \tilde{y}_n & 0 & \cdots & 0 & \tilde{y}_2 \\ 0 & 0 & \tilde{y}_n & \cdots & 0 & \tilde{y}_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \tilde{y}_n & \tilde{y}_{n-1} \\ -\tilde{y}_n & -\tilde{y}_n & -\tilde{y}_n & \cdots & -\tilde{y}_n & 1 - \sum_{i=1}^{n-1} \tilde{y}_i \end{vmatrix}$$

$$= \tilde{y}_n^{n-1} \tag{3.28}$$

as can be seen by adding the sum of the first n-1 rows to the last row. Application of the transformation (3.26) (that is, substituting equations (3.27) into (3.25) and multiplying by $|J| = \tilde{y}_n^{n-1}$) yields, upon rearranging, the j.p.d.f. of the vector random variable $\tilde{y} = (\tilde{y}_1,\ldots,\tilde{y}_n)$,

$$g_y(y_1,\ldots,y_n) = \frac{\prod_{i=1}^{n-1} y_i^{\alpha_i - 1}}{\prod_{i=1}^{n} \Gamma(\alpha_i)} \left(1 - \sum_{j=1}^{n-1} y_j\right)^{\alpha_j - 1} y_n^{\sum_{i=1}^{n} \alpha_i - 1} e^{-y_n} \tag{3.29}$$

$$\left(y_i \geq 0,\ i = 1,\ldots,n;\ \sum_{i=1}^{n-1} y_i \leq 1\right).$$

Upon integrating (3.29) with respect to y_n (recognizing that $\tilde{y}_n \sim \Gamma(\sum_{i=1}^{n} \alpha_i, 1)$) we have the j.p.d.f. of the normalized principal right

eigenvector; that is,

$$g_y(y_1,\ldots,y_{n-1}) = \frac{\Gamma\left(\sum_{i=1}^{n} \alpha_i\right)}{\prod_{i=1}^{n} \Gamma(\alpha_i)} \prod_{j=1}^{n-1} y_j^{\alpha_j - 1} \left(1 - \sum_{i=1}^{n-1} y_i\right)^{\alpha_n - 1} \tag{3.30}$$

at any point in the simplex: S_{n-1}: $\{(y_1,\ldots,y_{n-1})$: $y_i > 0$ for $i = 1,\ldots,n-1$, and $\sum_{i=1}^{n-1} y_i \leq 1\}$, in R_{n-1} and zero outside, and where α_i are all real and positive. Wilks (1962, p. 178) refers to a distribution having the j.p.d.f. (3.30) as the (n-1)-variate Dirichlet distribution $D(\alpha_1,\ldots,\alpha_{n-1}; \alpha_n)$.

As mentioned in Chapter II, any normalized column of a consistent pairwise comparison matrix yields the priority vector of the elements under comparison. Thus, the distribution of the normalized principal right eigenvector can be derived from the n^{th} column vector of the dominance matrix $\tilde{X}$ by applying the transformation

$$T: \tilde{y}_i = \frac{\tilde{x}_{in}}{\tilde{y}_n}, \quad i = 1,\ldots,n-1; \quad \tilde{y}_n = 1 + \sum_{j=1}^{n-1} \tilde{x}_{jn} \tag{3.31}$$

so that

$$T^{-1}: \tilde{x}_{in} = \tilde{y}_i\tilde{y}_n, \quad i = 1,\ldots,n-1; \quad \tilde{y}_n = \frac{1}{1 - \sum_{j=1}^{n-1} \tilde{y}_j}$$

or

$$\tilde{x}_{in} = \frac{\tilde{y}_i}{\left(1 - \sum_{j=1}^{n-1} \tilde{y}_j\right)} \qquad i = 1,\ldots,n-1. \tag{3.32}$$

The Jacobian of this transformation is

$$J = \frac{\partial(\tilde{x}_{1n},\ldots,\tilde{x}_{(n-1)n})}{\partial(\tilde{y}_1,\ldots,\tilde{y}_{n-1})}$$

$$J = \begin{vmatrix} \frac{\tilde{y}_1+k}{k^2} & \frac{\tilde{y}_1}{k^2} & \cdots & \frac{\tilde{y}_1}{k^2} \\ \frac{\tilde{y}_2}{k^2} & \frac{\tilde{y}_2+k}{k^2} & \cdots & \frac{\tilde{y}_2}{k^2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\tilde{y}_{n-1}}{k^2} & \frac{\tilde{y}_{n-1}}{k^2} & \cdots & \frac{\tilde{y}_{n-1}+k}{k^2} \end{vmatrix}$$

or

$$J = (\frac{1}{k^2})^{n-1} \begin{vmatrix} \tilde{y}_1+k & \tilde{y}_1 & \cdots & \tilde{y}_1 \\ \tilde{y}_2 & \tilde{y}_2+k & \cdots & \tilde{y}_2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \tilde{y}_{n-1} & \tilde{y}_{n-1} & \cdots & \tilde{y}_{n-1}+k \end{vmatrix} = (\frac{1}{k^2})^{n-1} |A| \qquad (3.34)$$

where $k = 1 - \sum_{j=1}^{n-1} \tilde{y}_j$. The matrix A can be written as

$$A = kI_{n-1} + 1_{n-1}^t \tilde{y} \qquad (3.35)$$

where

I_{n-1} is the identity matrix of order n-1,

1_{n-1} is a (n-1) x 1 column vector of 1's, and

$\tilde{y} = (\tilde{y}_1,\ldots,\tilde{y}_{n-1})^t$.

Using a variation of a result given in Aitken (1956, p. 133), the determinant of A is

$$|A| = \left|k I_{n-1} + 1_{n-1}^{t}\tilde{y}\right| = k^{n-1} + (\tilde{y}^{t}1_{n-1})k^{n-2} \tag{3.36}$$

or

$$|A| = k^{n-2}(k + \tilde{y}^{t}1_{n-1}) = k^{n-2}. \tag{3.37}$$

Upon substituting equation (3.37) into (3.34) we see that the Jacobian of the transformation is

$$|J| = (\tfrac{1}{k})^{n} = \left(\frac{1}{1 - \sum_{j=1}^{n-1} \tilde{y}_j}\right)^{n}. \tag{3.38}$$

Upon applying the transformation (that is, substituting equations (3.32) into (3.12) and multiplying by (3.38)) we have

$$g_y(y_1,\ldots,y_{n-1}) = \frac{\Gamma(\sum_{i=1}^{n} \alpha_i)}{\prod_{i=1}^{n} \Gamma(\alpha_i)} \prod_{j=1}^{n-1} \left(\frac{y_j}{1 - \sum_{k=1}^{n-1} y_k}\right)^{\alpha_j - 1} (y_n)^{-\sum_{j=1}^{n} \alpha_j} \left(\frac{1}{1 - \sum_{j=1}^{n-1} y_j}\right)^{n}$$

$$(y_i \geq 0,\ i = 1,\ldots,n-1;\ \sum_{j=1}^{n-1} y_j \leq 1), \tag{3.39}$$

where

$$\tilde{y}_n = \frac{1}{1 - \sum_{j=1}^{n-1} \tilde{y}_j}. \tag{3.40}$$

After substituting (3.40) into (3.39) we have the j.p.d.f. of the vector

random variable $\tilde{y} = (\tilde{y}_1,\ldots,\tilde{y}_{n-1})$; that is, the density of the normalized principal right eigenvector as given by density (3.30).

The general moment $\mu'_{r_1\ldots r_{n-1}}$ of the (n-1)-variate Dirichlet distribution has the value

$$\mu'_{r_1\ldots r_{n-1}} = \frac{\Gamma(\alpha_1 + r_1) \ldots \Gamma(\alpha_{n-1} + r_{n-1})\Gamma(\alpha_1 + \ldots + \alpha_n)}{\Gamma(\alpha_1 + \ldots + \alpha_n + r_1 \ldots + r_{n-1})\Gamma(\alpha_1) \ldots \Gamma(\alpha_n)} \tag{3.41}$$

from which the mean, variances, and the covariances of the y_i's are found to be

$$E[\tilde{y}_i] = \frac{\alpha_i}{\sum_{j=1}^{n} \alpha_j}, \qquad i = 1,\ldots,n-1 \tag{3.42}$$

$$V[\tilde{y}_i] = \frac{\alpha_i(\sum_{j=1}^{n} \alpha_j - \alpha_i)}{(\sum_{j=1}^{n} \alpha_j)^2(\sum_{j=1}^{n} \alpha_j + 1)}, \qquad i = 1,\ldots,n-1 \tag{3.43}$$

$$\mathrm{Cov}(\tilde{y}_i,\tilde{y}_j) = -\frac{\alpha_i\alpha_j}{(\sum_{k=1}^{n} \alpha_k)^2(\sum_{k=1}^{n} \alpha_k + 1)}, \qquad i \neq j = 1,\ldots,n-1. \tag{3.44}$$

An important property of the (n-1)-variate Dirichlet distribution is[2]

Property 3.2. If $(\tilde{y}_1,\ldots,\tilde{y}_{n-1})$ is a vector random variable having the (n-1)-variate Dirichlet distribution $D(\alpha_1,\ldots,\alpha_{n-1}; \alpha_n)$, then the marginal distribution of $(\tilde{y}_1,\ldots,\tilde{y}_k)$, $k < n-1$, is the k-variate Dirichlet distribution $D(\alpha_1,\ldots, {}_k; \alpha_{k+1} + \ldots + \alpha_n)$.

It follows from this property that the $\tilde{y}_i$, $i \neq n$, are marginally distributed as beta variates with parameters α_i and $\sum_{\substack{j=1 \\ i \neq j}}^{n} \alpha_j$, respectively, and with density

$$g_i(y_i) = \begin{cases} \dfrac{\Gamma(\sum_{i=1}^{n} \alpha_i)}{\Gamma(\alpha_i)\Gamma(\sum_{\substack{j=1 \\ i \neq j}}^{n} \alpha_j)} y_i^{\alpha_i - 1} (1 - y_i)^{\sum_{\substack{j=1 \\ i \neq j}}^{n} \alpha_j - 1}, & 0 < y_i \leq 1 \\ 0, & \text{elsewhere} \end{cases} \tag{3.45}$$

for $i = 1,\ldots,n-1$.

The mean and variance of the random variable y_i are given in equations (3.12) and (3.42), respectively.

As mentioned earlier, the m.p.d.f.'s of the $\tilde{y}_i$'s represent the priority distributions of the relevant decision elements in accordance with their impact upon the criterion of interest. With this in mind, it is useful as well as informative to interpret equations (3.42) and (3.43) in the context of the AHP paradigm. For illustrative purposes, consider the case of two decision elements i and j. Let the dominance ratio random variable $\tilde{x}_{ij}$ by distributed as an inverted beta-2 variate with parameters α_i, α_j; that is, $\tilde{x}_{ij} \sim i\beta_2(\alpha_i,\alpha_j)$. Then the priority of element i, $\tilde{y}_i$, is distributed as a beta variate with parameters α_i, α_j. If we assume that $\alpha_j >> 2$, the expected value and variance of the priority distribution over $\tilde{y}_i$ can be written, respectively, as

$$E(\tilde{y}_i) = \frac{\alpha_i}{\alpha_i + \alpha_j} \tag{3.46}$$

$$V(\tilde{y}_i) = \frac{\alpha_i \alpha_j}{(\alpha_i + \alpha_j)^3} \cdot \tag{3.47}$$

If $\alpha_i >> \alpha_j$, it is no surprise that the $E(\tilde{y}_i) \to 1$ and the $V(\tilde{y}_i) \to 0$ as $\alpha_i \to \infty$. Thus, the beta distribution over y_i becomes skewed towards the right as the magnitude of α_i increases relative to α_j, indicating that element i is receiving more and more priority weight relative to element j. A more interesting situation occurs if $\alpha_i = \alpha_j = \alpha >> 2$, in which case equations (3.46) and (3.47) can be written, respectively, as (after dropping the $\tilde{y}$ subscript)

$$E(\tilde{y}) = \frac{1}{2} \tag{3.48}$$

$$V(\tilde{y}) = \frac{1}{8\alpha} \cdot \tag{3.49}$$

Note that the variance of $\tilde{y}$ depends upon the magnitude of the importance parameter α. This implies that if a decision agent is indifferent (under uncertainty) between two decision elements, the variance of the priority distribution over the elements compared depends upon the scale of their absolute importance. The larger (smaller) the scale, the smaller (larger) the variance. Obviously this has some interesting behavioral implications and suggests a testable hypothesis concerning whether or not indifference perceptions under uncertainty are a function of the absolute importances of the elements compared. This facet of the current research will be investigated in future research efforts.

Another important property of the (n-1)-variate Dirichlet distribution can be stated as follows:[3]

Property 3.3. If $(\tilde{y}_1,\ldots,\tilde{y}_{n-1})$ is a vector random variable having the (n-1)-variate Dirichlet distribution $D(\alpha_1,\ldots,\alpha_{n-1}; \alpha_n)$, then the random variable $(\tilde{z}_1,\ldots,\tilde{z}_s)$, where $\tilde{z}_1 = \tilde{x}_1 + \ldots + \tilde{x}_{k_1}$, $\tilde{z}_2 = \tilde{x}_{k_1+1} + \ldots + \tilde{x}_{k_1+k_2}, \ldots, \tilde{z}_s = \tilde{x}_{k_1+\ldots+k_{s-1}+1} + \ldots + \tilde{x}_{k_1+\ldots+k_s}$, and $k_1 + \ldots + k_s \leq n-1$, has the s-variate Dirichlet distribution $D(\alpha_{(1)},\ldots,\alpha_{(s)}; \alpha_{(s+1)})$, where $\alpha_{(1)} = \alpha_1 + \ldots + \alpha_{k_1}, \ldots, \alpha_{(s)} = \alpha_{k_1+\ldots+k_{s-1}} + \ldots + \alpha_{k_1+\ldots+k_s}$, and $\alpha_{(s+1)} = \alpha_{k_1+\ldots+k_s+1} + \ldots + \alpha_n$.

This property is extremely important in the context of the AHP paradigm because of its usefulness in partitioning and aggregating a set of n decision elements into s subsets ($s \leq n$). The significance of clustering decision elements into groups is demonstrated in a subsequent chapter.

Notes

[1]Refer to Tiao and Guttman (1965), p. 795.

[2]Refer to Wilks (1964), pp. 179-180.

[3]Refer to Wilks (1964), p. 181.

[4]Refer to Saaty (1980), p. 78, for a discussion of clustering as a technique for modeling and analyzing complex hierarchically structured decision problems.

CHAPTER IV
EXTENSIONS TO THE PROBABILISTIC MODEL FOR EIGENVECTOR SCALING OF CONSISTENT DOMINANCE RATIOS

Introduction

The previous chapter developed the priority distribution for each of n decision elements according to their impact, under uncertainty, upon a governing criterion. This presentation helps provide a description of how the priority distributions of the various elements capture the uncertainty inherent in establishing the dominance relationships. Although much was learned about the specific effects of uncertainty upon the priority distributions, nothing was learned about the ultimate rankings of the decision elements. The purpose of this chapter is to develop a specific statement about such rankings; namely, the probability that one of the n decision elements will receive the highest ranking, given the parameters characterizing the uncertain pairwise comparison judgments.

The above probabilistic statement is of great importance, especially from the standpoint of the investigative agent mentioned in the previous chapter. Two situations that are of significant interest to such an agent come to the forefront. Both situations presume the existence of a surrogate decision model that simulates an actual or imagined decision model of another agent. Consider first the requirement of predicting (probabilistically) which of the n available decision

elements the other agent will choose given (a) past observed behavior of the agent in similar circumstances, (b) public information, or (c) inferential information either subjectively assessed, or obtained from experts or possibly from the other decision agent (as perhaps in the case of a cooperative policy-making body willing to supply rough measures or estimates of their judgments without revealing precisely their perhaps uncertain preferences). The second situation involves the requirement to better understand which elements of the decision problem predominate the selection of one of the several decision elements as the most preferred. This is a past decision-making investigative process. In this situation, it is of interest to determine the range of possible values of the parameters characterizing the uncertain dominance judgments necessary for the substantiation of the observed outcome. It may occur, in the case of policy-making bodies, that the observed outcome cannot be substantiated in terms of any enunciated objectives or relevant decision criteria set forth at the outset by the body. In this regard, it is essential that policy-making bodies come to better understand its decision-making processes and thereby make better future decisions. Perhaps the AHP paradigm (under certainty or uncertainty) may be of use in this regard. This is not to say that policy-making bodies should use formal decision models per se, but they should at least be aware of which decision elements require greater or lesser attention or scrutiny. In both the above situations, sensitivity analysis of the characterizing judgment parameters is extremely useful in establishing the robustness or stability of the ultimate ranking of the alternatives. The upshot of this is that the process may give

significant insight into which decision elements are important or unimportant to the decision at hand.

The following sections of this chapter address the problem of developing an expression for the probability that the i^{th} member of a set of n decision elements receives the highest ranking given the parameters characterizing the pairwise comparison matrix. This probabilistic expression is developed from both the principal right eigenvector and the j^{th} column vector of the pairwise comparison matrix. The latter development establishes a computational formula useful in evaluating the desired probability, while the former development establishes a simpler more concise equivalent expression that has theoretical interest. A computer program would be useful for calculating the probability that a specified decision element is optimal given the parameters $\alpha_1,\ldots,\alpha_n$ of the pairwise comparison process. Such a program would also provide a means for analyzing the robustness or stability of the highest priority ranking as the α-parameters are allowed to vary through specified ranges. This facet of the current research will be investigated in future research efforts.

The Probability that the i^{th} Decision Element Receives the Highest Priority Ranking

A natural place to begin developing an expression for the probability that a given element receives the highest priority ranking is the priority vector; that is, the normalized principal right eigenvector $\bar{y} = (\bar{y}_1,\ldots,\bar{y}_{n-1})$. Since the random variable y_i represents the priority weight assigned to decision element i, i = 1,...,n, the probability that element i receives the highest ranking is equivalent to the

probability that it has the greatest priority weight; that is,

$$P\left[\left(\bigcap_{\substack{j=1 \\ j\neq i}}^{n} \tilde{y}_j < \tilde{y}_i\right) \cap \left(\sum_{j=1}^{n-1} \tilde{y}_j < \tilde{y}_i\right)\right].$$

This probability is difficult to evaluate, however, because the joint distribution of the vector random variable $\tilde{y} = (\tilde{y}_1,\ldots,\tilde{y}_{n-1})$ (that is, the (n-1)-variate Dirichlet distribution $D(\alpha_1,\ldots,\alpha_{n-1};\ \alpha_n)$) is constrained to satisfy the relationship $\sum_{i=1}^{n-1} \tilde{y}_i \leq 1$, where $\tilde{y}_i \geq 0$, $i = 1,\ldots,n$. This simplex constraint makes it difficult to establish the limits of integration of

$$P\left[\left(\bigcap_{\substack{j=1 \\ j\neq i}}^{n} \tilde{y}_j < \tilde{y}_i\right) \cap \left(\sum_{j=1}^{n-1} \tilde{y}_j < \tilde{y}_i\right)\right]$$

for even small n. Equivalent probabilistic statements concerning element i's top ranking can be derived either from the principal right eigenvector $\tilde{v} = (\tilde{v}_1,\ldots,\tilde{v}_n)$ or from the n^{th} column vector $\tilde{x}_n = (\tilde{x}_{1n},\ldots,\tilde{x}_{(n-1)n})$ of the dominance matrix (see Tiao and Guttman, 1965). In the former case we are interested in the probability

$$P\left[\bigcup_{\substack{j=1 \\ j\neq i}}^{n} \tilde{v}_j < \tilde{v}_i\right]$$

and in the latter case the probability

$$P\left[\left(\bigcap_{\substack{j=1 \\ j\neq i}}^{n-1} \tilde{v}_j/\tilde{v}_n < \tilde{v}_i/\tilde{v}_n\right) \cap (\tilde{v}_1/\tilde{v}_n > 1)\right].$$

This section develops the probabilistic statement from the principal right eigenvector, leaving the alternative development for the next section.

Let the vector random variable $\tilde{v} = (\tilde{v}_1,\ldots,\tilde{v}_n)$ be the principal right eigenvector of the pairwise comparison matrix. As in previous discussions, assume that the $\tilde{v}_j$ are distributed as independent gamma variates with parameters α_j, and β; that is, $\tilde{v}_j \sim \Gamma(\alpha_j,\beta)$, $j = 1,\ldots,n$, with density

$$f_j(v_j) = \begin{cases} \dfrac{1}{\beta^{\alpha_j}\Gamma(\alpha_j)} v_j^{\alpha_j-1} e^{-v_j/\beta}, & v_j > 0 \quad (\alpha_j,\ \beta > 0) \\ 0, & \text{elsewhere.} \end{cases} \tag{4.1}$$

If the parameters characterizing the distributions of the $\tilde{v}_j$ are known, then the probability that the i^{th} element (of a set of n decision elements) receives the highest priority ranking can be found by evaluating the probability

$$P\Big(\bigcap_{\substack{j=1 \\ j\neq i}}^{n} \tilde{v}_j < \tilde{v}_i\Big).$$

Although the $\tilde{v}_j$ are independently distributed random variables, the events $\tilde{v}_j < \tilde{v}_i$, $j \neq i$, are not jointly independent events. Thus, the probability

$$P\Big(\bigcap_{\substack{j=1 \\ j\neq i}}^{n} \tilde{v}_j < \tilde{v}_i\Big)$$

must be computed by conditioning upon the random variable $\tilde{v}_i$. For notational convenience (and without loss of generality) let i = 1. Hence, we have upon conditioning with respect to $\tilde{v}_1$

$$P\left(\bigcap_{j=2}^{n} \tilde{v}_j < \tilde{v}_1\right) = \int_{\tilde{v}_1} P\left(\bigcap_{j=2}^{n} \tilde{v}_j < \tilde{v}_1 \middle| \tilde{v}_1 = v_1\right) dF_{\tilde{v}_1}(v_1). \tag{4.2}$$

The joint events $\tilde{v}_j < \tilde{v}_1$ are conditionally independent given $\tilde{v}_1 = v_1$; thus the probability $P\left(\bigcap_{j=2}^{n} \tilde{v}_j < \tilde{v}_1 \middle| \tilde{v}_1 = v_1\right)$ can be factored, and equation (4.2) can be written as

$$P\left(\bigcap_{j=2}^{n} \tilde{v}_j < \tilde{v}_1\right) = \int_{\tilde{v}_1} \prod_{j=2}^{n} P(\tilde{v}_j < \tilde{v}_1 \mid \tilde{v}_1 = v_1) dF_{\tilde{v}_1}(v_1) \tag{4.3}$$

or equivalently

$$P\left(\bigcap_{j=2}^{n} \tilde{v}_j < \tilde{v}_1\right) = \int_{\tilde{v}_1} \prod_{j=2}^{n} P(\tilde{v}_j < v_1) dF_{\tilde{v}_1}(v_1). \tag{4.4}$$

Further,

$$P(\tilde{v}_j < v_1) = \int_0^{v_1} dF_{\tilde{v}_j}(v_j), \qquad j = 2,\ldots,n \tag{4.5}$$

or

$$P(\tilde{v}_j < v_1) = \int_0^{v_1} \frac{v_j^{\alpha_j - 1} e^{-v_j/\beta}}{\beta^{\alpha_j} \Gamma(\alpha_j)} dv_j, \qquad j = 2,\ldots,n. \tag{4.6}$$

Thus, the integrand of (4.4) is, in part, the product of (n-1) incomplete gamma functions. Raiffa and Schlaifer (1961, p. 224) have shown

after successively integrating (4.6) by parts (providing that $\alpha_k > 0$ and integral) that

$$P(\tilde{v}_j < v_1) = \sum_{k=\alpha_j}^{\infty} \frac{e^{-v_1/\beta}(v_1/\beta)^k}{k!} = G_p(\alpha_j \mid v_1/\beta) \tag{4.7}$$

where $G_p(\alpha_j \mid v_1/\beta)$ is the right-tail cumulative function (see Raiffa and Schlaifer, 1961, p. 211) of a Poisson distributed random variable with parameter $\lambda = v_1/\beta$. It is convenient to define a new index of summation for equation (4.7); specifically, let $k_j = k - \alpha_j$, $j = 2,\ldots,n$. Hence, equation (4.7) is equivalently written as

$$P(\tilde{v}_j < v_1) = \sum_{k_j=0}^{\infty} \frac{e^{-v_1/\beta}(v_1/\beta)^{k_j+\alpha_j}}{(\alpha_j + k_j)!}, \qquad j = 2,\ldots,n. \tag{4.8}$$

Substituting equation (4.8) into equation (4.4) yields

$$P\left(\bigcap_{j=2}^{n} \tilde{v}_j < \tilde{v}_1\right) = \int_0^{\infty} \prod_{j=2}^{n} \left(\sum_{k_j=0}^{\infty} \frac{e^{-v_1/\beta}(v_1/\beta)^{k_j+\alpha_j}}{(\alpha_j + k_j)!} \right) \frac{e^{-v_1/\beta} v_1^{\alpha_1 - 1}}{\beta^{\alpha_1}\Gamma(\alpha_1)}\, dv_1. \tag{4.9}$$

If the power series given in (4.8) are absolutely convergent, then the series may be multiplied together term by term and rearranged to produce an absolutely convergent product series. To show that the power series (4.8) converges absolutely, consider that

$$\sum_{k_j=0}^{\infty} \left| \frac{e^{-v_1/\beta}(v_1/\beta)^{k_j}}{k_j!} \right| = 1 \ (< \infty) \tag{4.10}$$

and

$$\left|\frac{e^{-v_1/\beta}(v_1/\beta)^{k_j}}{(k_j+\alpha_j)!}\right| \le \left|\frac{e^{-v_1/\beta}(v_1/\beta)^{k_j}}{k_j!}\right| \tag{4.11}$$

hold for $j = 2,\ldots,n$. Thus, it follows that

$$\sum_{k_j=0}^{\infty}\left|\frac{e^{-v_1/\beta}(v_1/\beta)^{k_j}}{(k_j+\alpha_j)!}\right| \le \sum_{k_j=0}^{\infty}\left|\frac{e^{-v_1/\beta}(v_1/\beta)^{k_j}}{k_j!}\right|. \tag{4.12}$$

Hence, series (4.8) converges absolutely since it is dominated by an absolutely convergent series (that is, series (4.10)).

Using result B.10 (see Appendix B) equation (4.9) can be written as follows:

$$P\left(\bigcap_{j=2}^{n} \tilde{v}_j < \tilde{v}_1\right) =$$

$$\int_0^{\infty}\left[\sum_{k_n=0}^{\infty}\sum_{k_{n-1}=0}^{k_n}\cdots\sum_{k_2=0}^{k_3}\frac{e^{-(n-1)v_1/\beta}(v_1/\beta)^{\alpha_2+\ldots+\alpha_n+k_n}}{(\alpha_2+k_2)!\ \ldots\ (\alpha_n+k_n-k_{n-1})!}\right]\frac{e^{-v_1/\beta}v_1^{\alpha_1-1}}{\beta^{\alpha_1}\Gamma(\alpha_1)}\,dv_1. \tag{4.13}$$

Since the (n-1)-fold Cauchy product series appearing in the integrand of (4.13) is absolutely convergent, the order of integration and summation in equation (4.13) can be interchanged, yielding after rearranging,

$$P\left(\bigcap_{j=2}^{n} \tilde{v}_j < \tilde{v}_1\right) =$$

$$\sum_{k_n=0}^{\infty}\sum_{k_{n-1}=0}^{k_n}\cdots\sum_{k_2=0}^{k_3}\frac{(1/\beta)^{\alpha_1+\ldots+\alpha_n+k_n}}{(\alpha_1-1)!\,(\alpha_2+k_2)!\ldots(\alpha_n+k_n-k_{n-1})!}\int_0^{\infty}e^{-nv_1/\beta}$$

$$v_1^{\alpha_1+\ldots+\alpha_n+k_{n-1}}\,dv_1. \tag{4.14}$$

Integrating (4.14) with respect to v_1 (and recognizing that $v_1 \sim \Gamma(\alpha_1 + \ldots + \alpha_n + k_n, \beta/n)$) yields

$$P\left(\bigcap_{j=2}^{n} \tilde{v}_j < \tilde{v}_1\right) =$$

$$\sum_{k_n=0}^{\infty} \sum_{k_{n-1}=0}^{k_n} \cdots \sum_{k_2=0}^{k_3} \frac{(1/\beta)^{\alpha_1+\ldots+\alpha_n+k_n}}{(\alpha_1-1)!(\alpha_2+k_2)!\ldots(\alpha_n+k_n-k_{n-1})!}$$

$$\left(\frac{\beta}{n}\right)^{\alpha_1+\ldots+\alpha_n+k_n} \Gamma(\alpha_1+\ldots+\alpha_n+k_n). \tag{4.15}$$

After cancelling terms involving β and rearranging, equation (4.15) becomes

$$P\left(\bigcap_{j=2}^{n} \tilde{v}_j < \tilde{v}_1\right) =$$

$$\sum_{k_n=0}^{\infty} \sum_{k_{n-1}=0}^{k_n} \cdots \sum_{k_2=0}^{k_3} \binom{\alpha_1-1+\alpha_2+\ldots+\alpha_n+k_n}{\alpha_1-1 \;\; \alpha_2+k_2 \;\; \alpha_3+k_3-k_2 \ldots \alpha_n+k_n-k_{n-1}} \left(\frac{1}{n}\right)^{\alpha_1+\ldots+\alpha_n+k_n} \tag{4.16}$$

where

$$\binom{\alpha_1-1+\alpha_2+\ldots+\alpha_n+k_n}{\alpha_1-1 \;\; \alpha_2+k_2 \;\; \alpha_3+k_3-k_2 \ldots \alpha_n+k_n-k_{n-1}} =$$

$$\frac{(\alpha_1-1+\alpha_2+\ldots+\alpha_n+k_n)!}{(\alpha_1-1)!(\alpha_2+k_2)!(\alpha_3+k_3-k_2)!\ldots(\alpha_n+k_n-k_{n-1})!}. \tag{4.17}$$

Thus, equation (4.16) represents the probability that any one of the n decision elements (after appropriately reordering the subscripts) will receive the greatest priority weight, given the parameters characterizing the uncertain relative importance pairwise comparison judgments. Unfortunately, equation (4.16) is not very useful as computational formula; however, it does have the clear advantage of being a concise mathematical representation of the probability

$$P\Big(\bigcap_{\substack{j=1\\ j\neq i}}^{n} \tilde{v}_j < \tilde{v}_i\Big), \qquad i = 1,\ldots,n.$$

The next section is devoted to developing a useful computational algorithm for calculating the probability that any specified decision element is optimal.

A Computational Formula for Calculating the Probability that the i^{th} Decision Element Receives the Highest Priority Weight

In the previous section we considered the evaluation of the probability

$$P\Big(\bigcap_{\substack{j=1\\ j\neq i}}^{n} \tilde{v}_j < \tilde{v}_1\Big),$$

where $\tilde{v} = (\tilde{v}_1,\ldots,\tilde{v}_n)$ is the principal right eigenvector. However, this probability can be rewritten as follows:

$$P\Big(\bigcap_{\substack{j=1\\ j\neq i}}^{n} \tilde{v}_j < \tilde{v}_i\Big) = P\Big[\Big(\bigcap_{\substack{j=1\\ j\neq i}}^{n-1} \tilde{v}_j/\tilde{v}_n < \tilde{v}_i/\tilde{v}_n\Big) \cap \big(\tilde{v}_i/\tilde{v}_n > 1\big)\Big] \qquad (4.17)$$

where the random variables $\tilde{v}_j/\tilde{v}_n$, $j \neq n$, are the elements of the n^{th} column vector of the pairwise comparison matrix X. Define these elements as the random variables $\tilde{z}_j = \tilde{v}_j/\tilde{v}_n$, $j \neq n$, where the n^{th} column vector is defined as $\tilde{z} = (\tilde{z}_1,\ldots,\tilde{z}_{n-1})$. With these definitions, the j.p.d.f. of the n^{th} column vector (see equation (3.12)) can be written as

$$h_z(z_1,\ldots,z_{n-1}) = \frac{\Gamma\left(\sum_{i=1}^{n} \alpha_i\right)}{\prod_{i=1}^{n} \Gamma(\alpha_i)} \prod_{j=1}^{n-1} z_j^{\alpha_j - 1} \left(1 + \sum_{i=1}^{n-1} z_i\right)^{-\sum_{i=1}^{n} \alpha_i} \qquad (4.18)$$

for $0 < z_j < \infty$, $j = 1,\ldots,n-1$, in R_{n-1} and zero elsewhere. Thus, probability (4.17) can be written (assuming that i = 1) as

$$P[\tilde{z}_2 < \tilde{z}_1,\ldots,\tilde{z}_n < \tilde{z}_1, \tilde{z}_1 > 1]$$

$$= \frac{\Gamma(\alpha_1+\ldots+\alpha_{n+1})}{\Gamma(\alpha_1)\ldots(\alpha_{n+1})} \int_1^{\infty} \int_0^{z_1} \cdots \int_0^{z_1} \frac{z_1^{\alpha_1-1} \ldots z_n^{\alpha_n-1}}{(1+z_1+\ldots+z_n)^{\alpha_1+\ldots+\alpha_{n+1}}} dz_n dz_{n-1} \ldots dz_1. \qquad (4.19)$$

This n-fold integration can be evaluated by using the following integration formula (see Tiao and Guttman, 1965, p. 796)

$$\int_0^a (1 + x + t)^{-(\alpha+n)} t^{n-1} dt =$$

$$B(n,\alpha)(1+z)^{-\alpha} - \sum_{j=0}^{n-1} \binom{n-1}{j} a^j (1 + x + a)^{-(\alpha+j)} B(\alpha+j,n-j) \qquad (4.20)$$

where n is a positive integer, and (x,α,a) are positive quantities and $B(p,q)(= \Gamma(p)\Gamma(q)/\Gamma(p+q)$ is the complete beta function.

Integrating (4.19) with respect to z_n using (4.20) yields

$$P[\tilde{z}_2 < \tilde{z}_1,\ldots,\tilde{z}_n < \tilde{z}_1,\tilde{z}_1 > 1]$$

$$= \frac{\Gamma(\alpha_1+\ldots+\alpha_{n+1})}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_{n+1})} \int_1^\infty \int_0^{z_1} \cdots \int_0^{z_1} z_1^{\alpha_1-1} \ldots z_{n-1}^{\alpha_{n-1}-1}$$

$$\left\{ \frac{B(\alpha_n,\alpha_1+\ldots\alpha_{n-1}+\alpha_{n+1})}{(1+z_1+\ldots+z_{n-1})^{\alpha_1+\ldots+\alpha_{n-1}+\alpha_{n+1}}} \right.$$

$$- \sum_{i_n=0}^{\alpha_n-1} \binom{\alpha_n-1}{i_n} z_1^{i_n} (1+2z_1+\ldots+z_{n-1})^{-(\alpha_1+i_n+\alpha_2+\ldots+\alpha_{n-1}+\alpha_{n+1})}$$

$$\left. B(\alpha_1+i_n+\alpha_2+\ldots+\alpha_{n-1}+\alpha_{n+1},\ \alpha_n-i_n) \right\} dz_{n-1}\cdots dz_1. \tag{4.21}$$

After expanding the terms involving $B(\cdot,\cdot)$ and $\binom{\alpha_n-1}{i_n}$ and upon multiplying by $\Gamma(\alpha_1+\ldots+\alpha_{n+1})/\Gamma(\alpha_1)\ldots\Gamma(\alpha_{n+1})$ equation (4.21) becomes

$$P[\tilde{z}_2 < \tilde{z}_1,\ldots,\tilde{z}_n < \tilde{z}_1,\tilde{z}_1 > 1]$$

$$= \frac{\Gamma(\alpha_1+\ldots+\alpha_{n-1}+\alpha_{n+1})}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_{n-1})\Gamma(\alpha_{n+1})} \int_1^\infty \int_0^{z_1} \cdots \int_0^{z_1} \frac{z_1^{\alpha_1-1} \ldots z_{n-1}^{\alpha_{n-1}-1}}{(1+z_1+\ldots+z_{n-1})^{\alpha_1+\ldots+\alpha_{n-1}+\alpha_{n+1}}}$$

$$dz_{n-1}\cdots dz_1 \tag{4.22a}$$

$$= \sum_{i_n=0}^{\alpha_n-1} \int_1^\infty \int_0^{z_1} \cdots \int_0^{z_1} \frac{\Gamma(\alpha_1+i_n+\alpha_2+\ldots+\alpha_{n-1}+\alpha_{n+1})}{i_n!\Gamma(\alpha_1)\ldots\Gamma(\alpha_{n-1})\Gamma(\alpha_{n+1})}$$

$$\frac{z_1^{\alpha_1+i_n-1} \ldots z_{n-1}^{\alpha_{n-1}-1}}{(1+2z_1+\ldots+z_{n-1})^{\alpha_1+i_n+\ldots+\alpha_{n-1}+\alpha_{n+1}}} dz_{n-1}\cdots dz_1 \tag{4.22b}$$

multiplying (4.22b) by $\Gamma(\alpha_1 + i_n)/\Gamma(\alpha_1 + i_n)$ and rearranging yields

$$P[\tilde{z}_2 < \tilde{z}_1, \ldots, \tilde{z}_n < \tilde{z}_1, \tilde{z}_1 > 1]$$

$$= \frac{\Gamma(\alpha_1+\ldots+\alpha_{n-1}+\alpha_{n+1})}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_{n-1})\Gamma(\alpha_{n+1})} \int_1^\infty \int_0^{z_1} \cdots \int_0^{z_1} \frac{z_1^{\alpha_1-1} \ldots z_{n-1}^{\alpha_{n-1}-1}}{(1+z_1+\ldots+z_{n-1})^{\alpha_1+\ldots+\alpha_{n-1}+\alpha_{n+1}}} \, dz_{n-1}\ldots dz_1 \tag{4.23a}$$

$$- \sum_{i_n=0}^{\alpha_n-1} \binom{\alpha_1+i_n-1}{\alpha_1-1} \frac{\Gamma(\alpha_1+i_n+\alpha_2+\ldots+\alpha_{n-1}+\alpha_{n+1})}{\Gamma(\alpha_1+i_n)\Gamma(\alpha_2)\ldots\Gamma(\alpha_{n-1})\Gamma(\alpha_{n+1})}$$

$$\int_1^\infty \int_0^{z_1} \cdots \int_0^{z_1} \frac{z_1^{\alpha_1+i_n-1} \ldots z_{n-1}^{\alpha_{n-1}-1}}{(1+2z_1+\ldots+z_{n-1})^{\alpha_1+i_n+\ldots+\alpha_{n-1}+\alpha_{n+1}}} \, dz_{n-1}\ldots dz_1. \tag{4.23b}$$

For notational convenience, let

$$P_n(1; \alpha_1, \ldots, \alpha_n; \alpha_{n+1})$$

$$= \frac{\Gamma(\alpha_1+\ldots+\alpha_{n+1})}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_{n+1})} \int_1^\infty \int_0^{z_1} \cdots \int_0^{z_1} \frac{z_1^{\alpha_1-1} \ldots z_n^{\alpha_n-1}}{(1+z_1+\ldots+z_n)^{\alpha_1+\ldots+\alpha_{n+1}}} \, dz_n\ldots dz_1 \tag{4.24}$$

and

$$P_{n-1}(1; \alpha_1, \ldots, \alpha_{n-1}; \alpha_{n+1})$$

$$= \frac{\Gamma(\alpha_1+\ldots+\alpha_{n-1}+\alpha_{n+1})}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_{n-1})\Gamma(\alpha_{n+1})} \int_1^\infty \int_0^{z_1} \cdots \int_0^{z_1} \frac{z_1^{\alpha_1-1} \ldots z_{n-1}^{\alpha_{n-1}-1}}{(1+z_1+\ldots+z_{n-1})^{\alpha_1+\ldots+\alpha_{n-1}+\alpha_{n+1}}} \, dz_{n-1}\ldots dz_1 \tag{4.25}$$

$$\left[\text{note that} \quad P_1(1;\, \alpha_1;\, \alpha_{n+1}) = \frac{\Gamma(\alpha_1+\alpha_{n+1})}{\Gamma(\alpha_1)\Gamma(\alpha_{n+1})} \int_1^\infty \frac{z_1^{\alpha_1-1}}{(1+z_1)^{\alpha_1+\alpha_{n+1}}}\, dz_1\right] \quad (4.26)$$

$$P_{n-1}(2;\, \alpha_1+i_1, \alpha_2, \ldots, \alpha_{n-1};\, \alpha_{n+1})$$

$$= \frac{\Gamma(\alpha_1+i_n, \alpha_2+\ldots+\alpha_{n-1}+\alpha_{n+1})}{\Gamma(\alpha_1+i_n)\Gamma(\alpha_2)\ldots\Gamma(\alpha_{n-1})\Gamma(\alpha_{n+1})}$$

$$\int_1^\infty \int_0^{z_1} \cdots \int_0^{z_1} \frac{z_1^{\alpha_1+i_n-1} \cdots z_{n-1}^{\alpha_{n-1}-1}}{(1+2z_1+\ldots+z_{n-1})^{\alpha_1+i_n+\ldots+\alpha_{n-1}+\alpha_{n+1}}}\, dz_{n-1}\cdots dz_1 \quad (4.27)$$

where the integers following the parenthesis indicate the coefficient of z_1 in the denominator of the integrand of equation (4.19) arising from successive applications of equation (4.20). Hence, equation (4.19) can be written as

$$P[z_2 < z_1, \ldots, z_n < z_1, z_1 > 1] = P_n(1;\, \alpha_1, \ldots, \alpha_n;\, \alpha_{n+1}) \quad (4.28)$$

and equations (4.23) can be written recursively as

$$P_n(1;\, \alpha_1, \ldots, \alpha_n;\, \alpha_{n+1}) = P_{n-1}(1;\, \alpha_1, \ldots, \alpha_{n-1};\, \alpha_{n+1})$$

$$- \sum_{i_n=0}^{\alpha_n-1} \binom{\alpha_1+i_n-1}{\alpha_1-1} P_{n-1}(2;\, \alpha_1+i_n, \alpha_2, \ldots, \alpha_{n-1}, \alpha_{n+1}). \quad (4.29)$$

The evaluation of (4.29) is stated in the following lemma which is proved in Appendix B.

Lemma 4.1. Let α_i, $i = 1,\ldots,n+1$, be positive integers, then

$$P_n(1; \alpha_1,\ldots,\alpha_n; \alpha_{n+1}) = P_1(1; \alpha_1; \alpha_{n+1}) - \sum_{n+1>i_{k_2}>1} S_{\alpha_1}(\alpha_{i_{k_2}})$$

$$- \sum_{n+1>i_{k_3}>i_{k_2}>1} S_{\alpha_1}(\alpha_{i_{k_3}},\alpha_{i_{k_2}})$$

$$- \sum_{n+1>i_{k_4}>i_{k_3}>i_{k_2}>1} S_{\alpha_1}(\alpha_{i_{k_4}},\alpha_{i_{k_3}},\alpha_{i_{k_2}})$$

$$+ \ldots + (-1)^{n+1} S_{\alpha_1}(\alpha_n,\ldots,\alpha_2) \qquad (4.30)$$

where

$$S_{\alpha_1}(\alpha_{i_{k_2}}) = \sum_{i_{k_2}=0}^{\alpha_{k_2}-1} \binom{\alpha_1+i_{k_2}-1}{\alpha_1-1} P_1(2; \alpha_1+i_{k_2}; \alpha_{n+1}), \qquad (4.31)$$

$$S_{\alpha_1}(\alpha_{i_{k_3}},\alpha_{i_{k_2}}) = \sum_{i_{k_3}=0}^{\alpha_{i_{k_3}}-1} \sum_{i_{k_2}=0}^{\alpha_{i_{k_2}}-1} \binom{\alpha_1+i_{k_3}+i_{k_2}-1}{\alpha_1-1,i_{k_3},i_{k_2}} P_1(1; \alpha_1+i_{k_3}+i_{k_2}; \alpha_{n+1}) \qquad (4.32)$$

for $n + 1 > k_3 > k_2 > 1$,

$$S_{\alpha_1}(\alpha_{i_{k_4}},\alpha_{i_{k_3}},\alpha_{i_{k_2}}) = \sum_{i_{k_4}=0}^{\alpha_{i_{k_4}}-1} \sum_{i_{k_3}=0}^{\alpha_{i_{k_3}}-1} \sum_{i_{k_2}=0}^{\alpha_{i_{k_2}}-1} \binom{\alpha_1+i_{k_4}+i_{k_3}+i_{k_2}+1}{\alpha_1-1,i_{k_4},i_{k_3},i_{k_2}}$$

$$\cdot P_1(3; \alpha_1+i_{k_4}+i_{k_3}+i_{k_2}; \alpha_{n+1}) \qquad (4.33)$$

for $n + 1 > k_4 > k_3 > k_2 > 1$,

$$S_{\alpha_1}(\alpha_n,\alpha_{n-1},\ldots,\alpha_2) = \sum_{i_n=0}^{\alpha_n-1} \sum_{i_{n-1}=0}^{\alpha_{n-1}-1} \cdots \sum_{i_2=0}^{\alpha_2-1} \binom{\alpha_1+i_n+i_{n-1}+\ldots+i_2}{\alpha_1-1,i_n,\ldots,i_2}$$

$$\cdot P_1(n-1;\ \alpha_1+i_n+\ldots+i_2;\ \alpha_{n+1}) \ , \qquad (4.34)$$

and

$$P_1(m;\ \alpha_1+i_{k_2}+\ldots+i_{k_{m-1}};\ \alpha_{n+1})$$

$$= \left(\frac{1}{m}\right)^{\alpha_1+i_{k_1}+\ldots+i_{k_{m-1}}} \sum_{i_1=0}^{\alpha_1-1} \binom{\alpha_{n+1}+i_1-1}{\alpha_{n+1}-1} \left(\frac{m}{m+1}\right)^{i_n} \left(\frac{1}{m+1}\right)^{\alpha_n+1} . \qquad (4.35)$$

In comparison to equation (4.16), equation (4.30) is extremely cumbersome and lacks the essential succinctness in representing the probability statement. However, equations (4.30) through (4.35) constitute a computational algorithm which is very useful in calculating the probability

$$P[(\bigcap_{\substack{j=1\\ j\neq 1}}^{n} \tilde{z}_1 < \tilde{z}_i) \cap (\tilde{z}_i > 1)],$$

given the parameters $\alpha_1,\ldots,\alpha_n$. These equations permit the exact determination of the desired probability (as given in equation (4.19) without resorting to approximation procedures (as is necessary in the application of equation (4.16)).

CHAPTER V
A PROBABILISTIC MODEL FOR ANALYZING HIERARCHICALLY STRUCTURED MULTI-CRITERION DECISION PROBLEMS

Introduction

Previous chapters have addressed several extensions of the AHP paradigm that permit systematic analysis of single criterion, multi-decision element, decision problems under uncertainty. The purpose of this chapter is to extend the probabilistic AHP paradigm to include complex multi-criterion decision problems described by multi-decision factor/element interrelationships across the dependent levels of a hierarchy.

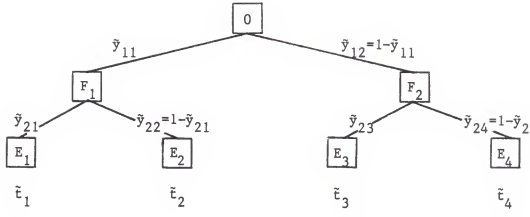
Probabilistic analysis of hierarchically structured multi-criterion, multi-decision element problems is analogous to the deterministic AHP analysis of such problems. The only conceptual difference is the treatment of the priority weights as random variables whose density functions represent priority distributions. Thus, each decision element in each level of the hierarchy is assigned a random variable that represents the priority weight allocated to that element in accordance with its relative strength with respect to those dominant criteria that it influences on the next higher level. The parameters characterizing the joint priority distributions are established in the construction of the distribution function of any column vector of the

pairwise comparison matrix of each dominant criterion. Using the principle of hierarchical composition the joint priority distribution function of the priority vector of the lowest-level elements with respect to the elements on the highest level of the hierarchy can be derived. Subsequently, the priority distribution function of each of the lowest level decision elements can be derived, and the relative strength or impact of these elements can be evaluated. In addition, the probability that a particular decision element at the lowest level of the hierarchy receives the greatest overall priority weight can also be derived and evaluated. What is significant about these results is that they are expressed in terms of the parameters characterizing all of the level-by-level individual element pairwise comparisons of the hierarchy.

Subsequent sections analyze the simplest of the hierarchically structured multi-criterion decision problems; those in which each element of each level dominates exactly two elements of the next lower level. Emphasis is placed upon binary branched hierarchies in order to reduce the number of random variables entering the model for multi-leveled hierarchies. Since the analytical technique is conceptually invariant to the number of levels and branching schema, the simplest illustrative model is presented.

Probabilistic Analysis of Priority Assignment in a Simple Hierarchically Structured Decision Problem

This section presents a probabilistic model for analyzing priority assignments in multi-criterion decision problems that are hierarchically



where

0--Overall Objective

F_1, F_2--Decision Factors

E_1, E_2, E_3, E_4--Decision Elements

y_{ij}--Priority weight assigned to element j of level i

t_i--Overall priority weight assigned to decision element E_i

Figure 5.1. A Simple Hierarchically Structured Decision Problem

modeled as depicted in Figure 5.1. The analysis is conducted by an investigative decision agent operating in an uncertain environment. Thus, the decision model may represent a known model that is used in the decision-making process of the decision agent under consideration, or may be a surrogate model that reflects the investigative agent's perceptions of the relevant decision criteria, elements, or factors of the problem faced by the decision agent in question. The hierarchy of Figure 5.1 represents a generic decision problem in which the goal is to attain an overall objective O by selecting between two independent decision factors, or factor clusters, F_1 and F_2. These factors are assumed to be influenced by independent factor-specific pairs of independent decision elements, (E_1,E_2) and (E_3,E_4), respectively. The purpose of this section is to derive the joint priority distribution of the elements E_1, E_2, E_3, and E_4 with respect to objective O, and to derive the probability

$$P[\bigcap_{\substack{j=1 \\ j\neq 1}}^{n} E_i > E_j \mid \alpha_1,\ldots,\alpha_n]$$

where the level-by-level pairwise comparison parameters are given by $\alpha_1,\ldots,\alpha_n$.

Suppose that the random variables $\tilde{y}_{11}$, $\tilde{y}_{21}$, and $\tilde{y}_{23}$ shown in Figure 5.1 represent the priority weights assigned to decision factor F_1, and to decision elements E_1 and E_3, respectively. These random variables are independent if and only if all of the elements within each level of the hierarchy are not interdependent. That is, there are no

relative importance interaction effects between subordinate elements within each level of the hierarchy. Consequently, the relative importance dominance judgments represent main effects, and the priority weight assigned to each element is a measure of that element's unique contribution to the criterion of interest. Hence, in the absence of interdependence, the priority weight random variables for the various criteria are independent. Having established this important philosophical and probabilistic argument, the decision problem of Figure 5.1 will now be analyzed step by step.

The first step in the analysis of any hierarchically structured problem using the AHP paradigm is the construction of the pairwise comparison matrix. Denoting the pairwise comparison matrix of the elements subordinate to criterion C as X(C), then

$$\tilde{X}(O) = \begin{array}{c} \\ F_1 \\ F_2 \end{array} \begin{array}{c} \begin{array}{cc} F_1 & F_2 \end{array} \\ \begin{bmatrix} 1 & \tilde{x}_{12} \\ \tilde{x}_{12}^{-1} & 1 \end{bmatrix} \end{array} \tag{5.1}$$

is the pairwise comparison matrix whose random variable entries indicate the strength with which decision factor F_i dominates factor F_j with respect to their influence upon objective O under uncertainty, $i \neq j = 1,2$. Likewise,

$$\tilde{X}(F_1) = \begin{array}{c} \\ E_1 \\ E_2 \end{array} \begin{array}{c} \begin{array}{cc} E_1 & E_2 \end{array} \\ \begin{bmatrix} 1 & \tilde{w}_{12} \\ \tilde{w}_{12}^{-1} & 1 \end{bmatrix} \end{array} \tag{5.2}$$

and

$$\tilde{X}(F_2) = \begin{array}{c} \\ E_3 \\ E_4 \end{array} \begin{array}{c} \begin{array}{cc} E_3 & E_4 \end{array} \\ \begin{bmatrix} 1 & \tilde{w}_{34} \\ \tilde{w}_{34}^{-1} & 1 \end{bmatrix} \end{array} \tag{5.3}$$

are the pairwise comparison matrices whose random variable entries indicate the strength with which decision element $E_i(E_{i'})$ dominates element $E_j(E_{j'})$ with respect to their influence upon decision factor $F_1(F_2)$, $i = j = 1,2$ $(i' \neq j' = 3,4)$.

Assume that the n^{th} $(n = 2)$ column vector of the pairwise comparison matrices $X(0)$, $X(F_1)$, and $X(F_2)$ are distributed as follows:

$$\tilde{x}_{12} \sim i\beta_2(\alpha_1,\alpha_2) \tag{5.4a}$$

$$\tilde{w}_{12} \sim i\beta_2(\gamma_1,\gamma_2) \tag{5.4b}$$

$$\tilde{w}_{34} \sim i\beta_2(\zeta_1,\zeta_2). \tag{5.4c}$$

(The distribution of the vector random variables of the $\tilde{X}(\cdot)$ are inverted beta-2 distributions since the $D'(\nu_1;\nu_2)$ is identical to the $i\beta_2(\nu_1,\nu_2)$, that is, density (3.12) reduces to the p.d.f. of an inverted beta-2 distribution for the case of $n - 1 = 1$). The parameters α_i, γ_i, and ζ_i, $i = 1,2$, characterizing distributions (5.4) are assumed to have been established using the elicitation proceedure advocated in Chapter III.

The next step in the analysis is to derive the normalized principal right eigenvector (priority vector) from the n^{th} column vector using transformation (3.31). Denoting the normalized principal right eigenvector of the elements subordinate to criterion C as $\tilde{y}(C)$, we have

$$\tilde{y}_1(0) = \begin{matrix} 0 \\ F_1 \\ F_2 \end{matrix} \begin{bmatrix} \tilde{y}_{11} \\ 1-\tilde{y}_{11} \end{bmatrix} \tag{5.5}$$

$$\tilde{y}_2(F_1) = \begin{matrix} F_1 \\ E_1 \\ E_2 \end{matrix} \begin{bmatrix} \tilde{y}_{21} \\ 1-\tilde{y}_{21} \end{bmatrix} \tag{5.6}$$

$$\tilde{y}_2(F_2) = \begin{matrix} F_2 \\ E_3 \\ E_4 \end{matrix} \begin{bmatrix} \tilde{y}_{23} \\ 1-\tilde{y}_{23} \end{bmatrix} \tag{5.7}$$

where

$$\tilde{y}_{11} \sim \beta(\alpha_1,\alpha_2) \tag{5.8a}$$

$$\tilde{y}_{21} \sim \beta(\gamma_1,\gamma_2) \tag{5.8b}$$

$$\tilde{y}_{23} \sim \beta(\zeta_1,\zeta_2) \tag{5.8c}$$

(since $D(\nu_1;\nu_2)$ is identical to $\beta(\nu_1,\nu_2)$, that is, density (3.30) reduces to density (3.45) for the case of $n - 1 = 1$).

Since $\tilde{y}_{11}$, $\tilde{y}_{21}$, $\tilde{y}_{23}$ are independent random variables, the j.p.d.f. of the vector random variable $\tilde{y} = (\tilde{y}_{11},\tilde{y}_{21},\tilde{y}_{23})$ can be uniquely determined by the product of the m.p.d.f.'s of distributions (5.8); that is

$$f_y(y_{11},y_{21},y_{23}) = \frac{y_{11}^{\alpha_1-1}(1-y_{11})^{\alpha_2-1}}{B(\alpha_1,\alpha_2)} \frac{y_{21}^{\gamma_1-1}(1-y_{21})^{\gamma_2-1}}{B(\gamma_1,\gamma_2)} \frac{y_{23}^{\zeta_1-1}(1-y_{23})^{\zeta_2-1}}{B(\zeta_1,\zeta_2)} \tag{5.9}$$

for $0 < y_{11} \leq 1$, $0 < y_{21} \leq 1$, and $0 < y_{23} \leq 1$, where $B(a,b) = \Gamma(a+b)/\Gamma(a)\Gamma(b)$.

Rearranging (5.9) yields

$$f_y(y_{11},y_{21},y_{23}) = K\, y_{11}^{\alpha_1-1}\, y_{21}^{\gamma_1-1}\, y_{23}^{\zeta_1-1}\, (1-y_{11})^{\alpha_2-1}\, (1-y_{21})^{\gamma_2-1}\, (1-y_{23})^{\zeta_2-1} \tag{5.10}$$

for $0 < y_{11} \leq 1$, $0 < y_{21} \leq 1$, and $0 < y_{23} \leq 1$, where $K = [B(\alpha_1, \alpha_2)B(\gamma_1, \gamma_2)B(\zeta_1, \zeta_2)]^{-1}$.

We next apply the principle of hierarchical composition to derive the priority vector of the lowest-level elements of the hierarchy with respect to the highest-level element. Using equation (2.9) with $h = 2$ (and denoting t for w and t' for w'), we have

$$\tilde{t} = \tilde{B}_2\tilde{t}' = \begin{bmatrix} \tilde{y}_2(F_1) & 0 \\ 0 & \tilde{y}_2(F_2) \end{bmatrix} \tilde{y}(0) \tag{5.11}$$

where $\tilde{B}_2$ is the composite priority matrix of the second level of the hierarchy, and $\tilde{y}_1(0)$, $\tilde{y}_2(F_1)$, and $\tilde{y}_2(F_2)$ are priority vectors defined by equations (5.5), (5.6), and (5.7), respectively. Hence, the priority vector $\tilde{t}$ is

$$\tilde{t} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \end{array} \begin{array}{c} \begin{array}{cc} F_1 & F_2 \end{array} \\ \begin{bmatrix} \tilde{y}_{21} & 0 \\ 1-\tilde{y}_{21} & 0 \\ 0 & \tilde{y}_{23} \\ 0 & 1-\tilde{y}_{23} \end{bmatrix} \end{array} \begin{array}{c} \begin{array}{c} \\ F_1 \\ F_2 \end{array} \\ \begin{bmatrix} \tilde{y}_{11} \\ 1-\tilde{y}_{11} \end{bmatrix} \end{array} \tag{5.12}$$

or

$$\underset{\tilde{t}}{} = \begin{matrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{matrix} \begin{bmatrix} \tilde{y}_{11}\tilde{y}_{21} \\ \tilde{y}_{11}(1-\tilde{y}_{21}) \\ (1-\tilde{y}_{11})\tilde{y}_{23} \\ (1-\tilde{y}_{11})(1-\tilde{y}_{23}) \end{bmatrix}. \qquad (5.13)$$

The components of the vector random variable t given in (5.13) can also be obtained by inspection of Figure 5.1. The composite priority weight of element E_i, that is $\tilde{t}_i$, can be obtained by multiplying together the priority weight random variables encountered along the path connecting the objective 0 to each decision element E_i, i = 1, 2, 3, 4. This alternative offers more intuitive appeal than the formal application of the principle of hierarchical composition; in either case the derived results are identical.

The joint priority distribution of the vector random variable $\tilde{t} = (\tilde{t}_1, \tilde{t}_2, \tilde{t}_3)$ can be obtained from density (5.10) by applying the transformation derived from the application of the principle of hierarchical composition, that is

$$T: \quad \tilde{t}_1 = \tilde{y}_{11}\tilde{y}_{21} \qquad (5.14a)$$

$$\tilde{t}_2 = \tilde{y}_{11}(1-\tilde{y}_{21}) \qquad (5.14b)$$

$$\tilde{t}_3 = (1-\tilde{y}_{11})\tilde{y}_{23} \qquad (5.14c)$$

$$\tilde{t}_4 = (1-\tilde{y}_{11})(1-\tilde{y}_{23}). \qquad (5.14d)$$

The following relationships are a consequence of equations (5.14):

$$\xi_1 + \xi_2 = \tilde{y}_{11} \tag{5.15a}$$

$$\xi_3 + \xi_4 - 1 - \tilde{y}_{11} = 1 - (\xi_1 + \xi_2) \tag{5.15b}$$

$$\xi_4 = 1 - (\xi_1 + \xi_2 + \xi_3) \tag{5.15c}$$

$$\sum_{j=1}^{3} \xi_j \leq 1 \ . \tag{5.15d}$$

The inverse of the transformation T is

$$T^{-1}: \quad \tilde{y}_1 = \xi_1 + \xi_2 \tag{5.16a}$$

$$\tilde{y}_2 = \frac{\xi_1}{\xi_1 + \xi_2} \tag{5.16b}$$

$$\xi_3 = \frac{\xi_3}{(1 - (\xi_1 + \xi_2))} \ . \tag{5.16c}$$

Hence, the Jacobian of T is

$$J = \frac{\partial(\tilde{y}_{11}, \tilde{y}_{21}, \tilde{y}_{23})}{\partial(\xi_1, \xi_2, \xi_3)} = \begin{vmatrix} 1 & 1 & 0 \\ \dfrac{\xi_2}{(\xi_1 + \xi_2)^2} & \dfrac{-\xi_1}{(\xi_1 + \xi_2)^2} & 0 \\ \dfrac{\xi_3}{(1-(\xi_1+\xi_2))^2} & \dfrac{\xi_3}{(1-(\xi_1+\xi_2))^2} & \dfrac{1}{(1-(\xi_1+\xi_2))} \end{vmatrix} \tag{5.17}$$

and

$$|J| = \frac{1}{(\xi_1 + \xi_2)(1 - (\xi_1 + \xi_2))} \ . \tag{5.18}$$

Application of the transformation T (that is, substituting equations (5.16) into density (5.10) and multiplying by (5.18)) yields the j.p.d.f.

of the vector random variable $\tilde{t} = (\tilde{t}_1, \tilde{t}_2, \tilde{t}_3)$, that is

$$g_t(t_1,t_2,t_3) = K(t_1+t_2)^{\alpha_1-1}\left(\frac{t_1}{t_1+t_2}\right)^{\gamma_1-1}\left(\frac{t_3}{1-(t_1+t_2)}\right)^{\zeta_1-1}(1-(t_1+t_2))^{\alpha_2-1}$$

$$\cdot \left(1-\frac{t_1}{t_1+t_2}\right)^{\gamma_2-1}\left(1-\frac{t_3}{1-(t_1+t_2)}\right)^{\zeta_2-1}\left(\frac{1}{(t_1+t_2)(1-(t_1+t_2))}\right) \tag{5.19}$$

for $t_i > 0$, $i = 1,2,3$, and $t_1 + t_2 + t_3 \leq 1$, and zero elsewhere.

After collecting like terms and rearranging, equation (5.19) can be written as

$$g_t(t_1,t_2,t_3) = K\, t_1^{\gamma_1-1}\, t_2^{\gamma_2-1}\, t_3^{\zeta_1-1}\,(1-(t_1+t_2+t_3))^{\zeta_2-1}$$

$$\cdot (t_1+t_2)^{\alpha_1-(\gamma_1+\gamma_2)}\,(1-(t_1+t_2))^{\alpha_2-(\zeta_1+\zeta_2)} \tag{5.20}$$

for $t_i > 0$, $i = 1,2,3$, and $t_1 + t_2 + t_3 \leq 1$, and zero elsewhere, with $K = [B(\alpha_1,\alpha_2)B(\gamma_1,\gamma_2)B(\zeta_1,\zeta_2)]^{-1}$.

Equation (5.20) is the joint priority distribution of the vector random variable $t = (t_1,t_2,t_3)$, where $\tilde{t}_i$ is the priority weight of decision element E_i with respect to objective 0, $i = 1,2,3$.

The trivariate density (5.20) is referred to by Antelman (1972) as the Dirichlet-beta density. In its general form, this density is quite difficult to work with; the specific problem is the difficulty of deriving the probability

$$P\Big[\bigcap_{\substack{j=1\\ j\neq i}}^{3} E_i > E_j \,\Big|\, \alpha_1,\ldots,\alpha_n\Big] = P\Big[\Big(\bigcap_{\substack{j=1\\ j\neq i}}^{3} \tilde{t}_j < \tilde{t}_i\Big) \cap \Big(\sum_{j=1}^{3} \tilde{t}_j < \tilde{t}_i\Big)\Big] \tag{5.21}$$

which is of great interest in the context of the probabilistic AHP paradigm. The Dirichlet-beta density can, however, be converted into a Dirichlet density by assuming that

$$\alpha_1 = \gamma_1 + \gamma_2 \tag{5.22a}$$

and

$$\alpha_2 = \zeta_1 + \zeta_2. \tag{5.22b}$$

Thus, substituting equations (5.22) into density (5.20) yields

$$g_t(t_1,t_2,t_3) = \frac{\Gamma(\gamma_1+\gamma_2+\zeta_1+\zeta_2)}{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\zeta_1)\Gamma(\zeta_2)}\, t_1^{\gamma_1-1}\, t_2^{\gamma_2-1}\, t_3^{\zeta_1-1}\, (1-(t_1+t_2+t_3))^{\zeta_2-1} \tag{5.23}$$

for any point in the simplex S_3: $\{(t_1,t_2,t_3)\colon t_i > 0,\ i = 1,2,3$ and $\sum_{i=1}^{3} t_i \leq 1\}$ in R_3, and zero elsewhere.

The m.p.d.f.'s of the $\tilde{t}_i$'s can be obtained by direct integration of density (5.23) or by utilizing property 3.2; in either case, the $\tilde{t}_i$ are distributed as beta variates,

$$\tilde{t}_1 \sim \beta(\gamma_1, \gamma_2+\zeta_1+\zeta_2) \tag{5.24a}$$

$$\tilde{t}_2 \sim \beta(\gamma_2, \gamma_1+\zeta_1+\zeta_2) \tag{5.24b}$$

$$\tilde{t}_3 \sim \beta(\zeta_1, \gamma_1+\gamma_2+\zeta_2) \tag{5.24c}$$

and

$$\tilde{t}_4 \sim \beta(\zeta_2, \gamma_1+\gamma_2+\zeta_1). \tag{5.24d}$$

Thus the assumptions of equations (5.22) are the necessary parameter relationships required to establish that the product of the independent beta distributed random variables of the transformation equations (5.14) be beta distributed. The assumptions of equations (5.22) cause the joint priority distribution of each level of the hierarchy to be Dirichlet distributed. Thus, not only are the distributions of the normalized principal right eigenvectors of the subordinate elements of each criterion Dirichlet distributed, but the joint density associated with all of the subordinate elements in each level of the hierarchy is also Dirichlet distributed. There is a price to be paid, however, for the benefits of tractable distributional analysis. In assuming that $\alpha_1 = \gamma_1 + \gamma_2$ and $\alpha_2 = \zeta_1 + \zeta_2$ we lose a degree of freedom in specifying the magnitude of the parameters which characterize the uncertain relative importance judgments between elements at the next lowest level of the hierarchy.

The exact nature of this loss of specification freedom can be found by examining the effects of the restrictions (5.22) upon the elicitation procedure. For illustrative purposes consider only the effects of the restriction $\alpha_1 = \gamma_1 + \gamma_2$ (only one of the restrictions need be considered since the overall effect on the elicitation process is the same for both restrictions). From equation (5.4b) we have that the distribution of the relative importance of decision element E_1 relative to element E_2 with respect to factor F_1, that is, $\tilde{w}_{12}$, is

distributed as an inverted beta-2 variate. Using equations (3.18) and (3.19) and assuming that $\alpha_2 >> 2$ we have

$$E(\tilde{w}_{12}) = \frac{\gamma_1}{\gamma_2} = n' \tag{5.25}$$

and

$$V(\tilde{w}_{12}) = \frac{n'(n' + 1)}{\gamma_2} = (\sigma')^2 \tag{5.26}$$

where n' and $(\sigma')^2$ are supplied by the decision agent. Substituting the restriction $\alpha_1 = \gamma_1 + \gamma_2$ into equation (5.25) and solving for γ_1 yields

$$\gamma_1 = \left(\frac{n'}{1 + n'}\right) \alpha_1. \tag{5.27}$$

The parameter α_1 is already known from the characterization of the distribution of $\tilde{x}_{12}$. Thus, the value of the parameter γ_1 follows immediately from (5.27) given the specification of n'. The value of γ_2 is fixed, since $\gamma_1 + \gamma_2 = \alpha_1$. Thus, the parameter restrictions result in the loss of freedom to specify γ_1 and γ_2, the parameters that characterize the distribution of the relative importance ratio $\tilde{w}_{12}$. The above analysis also suggests that equation (5.26) is redundant for elicitation purposes. Thus, the elicitation process for the lowest level of the hierarchy resembles that used in the deterministic AHP process.

The obvious question is why use the probabilistic approach when it reduces to a deterministic elicitation procedure? The answer is that the probabilistic model captures the uncertainty of the relative importance judgments at the highest level of the hierarchy, where the effects

of uncertainty are felt the most. It is at these levels where the articulation of preference ratios is the most difficult; hence, it is extremely useful if the uncertainty inherent in establishing the relative strength of the most important decision elements can be captured. This is preferable to assuming away the problem of uncertainty and resorting to a deterministic model. Assuredly, if the solution to a decision problem is "clear," it is unnecessary to employ any decision model. Another less philosophical but nonetheless compelling answer to the above question is that the probabilistic approach enables a hierarchically structured problem to be mathematically modeled in equation form, a virtue lacking in its deterministic counterpart. This feature gives the investigative decision agent a conceptual tool that can be used to test the effects of various parameter relationships or model assumptions. Taken together these features make the restricted parameter probabilistic model useful as a decision aid, albeit one that is less than perfect.

There remains one final step in the analysis of the above decision problem; namely, the derivation of the probability (5.21). Without loss of generality, let i = 1; hence probability (5.21) is written as

$$P[E_1 > E_2, E_1 > E_3, E_1 > E_4]$$

$$= P[\tilde{t}_1 > \tilde{t}_2, \tilde{t}_1 > \tilde{t}_3, \tilde{t}_1 > (1-(\tilde{t}_1+\tilde{t}_2+\tilde{t}_3))]. \tag{5.28}$$

The derivation of this probability is greatly simplified if the following transformation is applied to density (5.23):

$$T: \tilde{z}_i = \frac{\tilde{t}_i}{\tilde{z}_n}, \; i = 1,2,3; \qquad \tilde{z}_n = 1 - \sum_{i=1}^{3} \tilde{t}_i. \tag{5.29}$$

The inverse transformation is

$$T^{-1}: \quad \tilde{t}_i = \tilde{z}_i \tilde{z}_n, \; i = 1,2,3; \qquad \tilde{z}_n = \frac{1}{1 + \sum_{i=1}^{3} \tilde{z}_i}. \tag{5.30}$$

Using the results derived by Tiao and Guttman (1965, pp. 793-94) (see Appendix D) the j.p.d.f. of the vector random variable $\tilde{z} = (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3)$ is

$$h_z(z_1,z_2,z_3) = \frac{\Gamma(\gamma_1+\gamma_2+\zeta_1+\zeta_2)}{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\zeta_1)\Gamma(\zeta_2)} \frac{z_1^{\gamma_1-1} z_2^{\gamma_2-1} z_3^{\zeta_1-1}}{(1+Z_1+z_2+z_3)^{\gamma_1+\gamma_2+\zeta_z+\zeta_2}} \tag{5.31}$$

for $0 < z_i < \infty$, and zero elsewhere. Distribution (5.31) is the trivariate inverted Dirichlet distribution $D'(\gamma_1,\gamma_2,\zeta_1; \zeta_2)$. Thus, probability (5.28) can be written as

$$P[E_1 > E_2, E_1 > E_3, E_1 > E_4] = P[\tilde{z}_1 > \tilde{z}_2, \tilde{z}_1 > \tilde{z}_3, \tilde{z}_1 > 1]. \tag{5.32}$$

Probability (5.32) can be expressed in terms of the parameters γ_1, γ_2, ζ_1, and ζ_2 by using Lemma 4.1, that is

$$\begin{aligned} P[\tilde{z}_1 > \tilde{z}_2, \tilde{z}_1 > \tilde{z}_3, \tilde{z}_1 > 1] = & \sum_{i=0}^{\gamma_1-1} \binom{\zeta_2+i-1}{\zeta_2-1} \\ & - \sum_{i=0}^{\gamma_2-1} \binom{\gamma_1+i-1}{\gamma_1-1} \sum_{j=0}^{\gamma_1+i-1} \binom{\zeta_2+j-1}{\zeta_2-1} \left(\tfrac{1}{2}\right)^{\zeta_2+j} \\ & - \sum_{i=0}^{\zeta_1-1} \binom{\gamma_1+i-1}{\gamma_1-1} \left(\tfrac{1}{2}\right)^{\gamma_1+1} \sum_{j=0}^{\gamma_1+i-1} \binom{\zeta_2+j-1}{\zeta_2-1} \left(\tfrac{2}{3}\right)^{j} \left(\tfrac{1}{3}\right)^{\zeta_2} \\ & + \sum_{i=0}^{\zeta_1-1} \sum_{j=0}^{\gamma_2-1} \binom{\gamma_1+i+j-1}{\gamma_1,i,j} \left(\tfrac{1}{3}\right)^{\gamma_1+i+j} \sum_{k=0}^{\gamma_1+i+j-1} \binom{\zeta_2+k-1}{\zeta_2-1} \left(\tfrac{3}{4}\right)^{j} \left(\tfrac{1}{4}\right)^{\zeta_2}. \end{aligned} \tag{5.33}$$

Equation (5.33) can be solved using a computer program. By reordering the subscripts of (5.33), the probability that each of the E_i, in turn, is the most preferred element can be calculated. This analysis is useful from the standpoint of an investigative agent; specifically the selection of the element most likely to meet the overall objectives of the other decision agent under uncertainty, or the understanding of what the parameters of the model must be if a given element is observed to be selected as the most preferred. This will be the subject of future research.

CHAPTER VI
PROBABILISTIC ANALYSIS OF THE DECISION OF THE BOARD OF REGENTS OF THE UNIVERSITY OF OKLAHOMA ET AL. TO BRING SUIT AGAINST THE NATIONAL COLLEGIATE ATHLETIC ASSOCIATION

Introduction

In June, 1984, the United States Supreme Court, by a vote of seven to two, agreed with two lower Federal courts that the controls exercised by the National Collegiate Athletic Association (NCAA) over the television broadcasting rights of its member schools, as set forth in the "1982-85 NCAA Television Plan," violate the Sherman Antitrust Act. Consequently, contracts for live telecasts of the NCAA football games totaling some 280 million dollars over the 1982-85 playing seasons were invalidated. As a result, universities and colleges are free to negotiate autonomously for the assignment of television broadcasting rights for their football games. It is notable that these significant events were triggered by the actions of only two of the 506 NCAA member schools that play intercollegiate varsity football, although sixty of the Division I NCAA members actively sought to change the television control politics. The purpose of this chapter is to conduct a decision analysis investigation of the rationality of the plaintiff's action to bring suit against the NCAA. The objective of the investigation is twofold: (1) to structure the decision problem hierarchically in terms of general economic and non-economic decision criteria and factors; and (2) to probabilistically analyze the assignment of

priorities over the decision criteria and factors by varying the perameters characterizing the priority distributions. This investigative process is a highly flexible tool for systematically exploring the effects of a wide range of relative importance preference judgments upon the selection of the best activity or course of action. The purpose of the analysis is to learn more about the decision-making processes of collegiate athletic decision agents under uncertainty. The upshot is to draw inferences about various combinations of priority assignments that substantiate the observed actions taken by the decision agents; the underlying goal being to gain insight into why so few decision agents opted to take legal action against the NCAA.

Background

The NCAA is a private non-profit association, whose membership consists of approximately 780 colleges and universities. Members of the NCAA are placed into one of three divisions depending upon the size of their athletic programs. Of its members, approximately five hundred field intercollegiate varsity football teams. The NCAA was established in 1905 for the purpose of regulating the conduct of intercollegiate athletic contests. The NCAA regulates a number of important aspects of collegiate sports, so as to organize play and to ensure the amateur status of college athletes. The regulations are consistent with the maintenance of the concept of athletics performed by "scholar athletes." The NCAA determines playing rules, sets eligibility requirements, regulates recruiting, and establishes the requirements for the number of scholarships schools may award. It

establishes when the playing seasons of intercollegiate sports may start and must end, determines the number of games that may be played, and fixes the number of coaches a team may have. These rules and regulations are reinforced by the threatened use of various sanctions. Among sanctions often levied for violation of these rules are the restriction of television appearances and banishment from participation in post-season playoffs or bowl games.

In the early 1950s, the NCAA enacted restrictions on the telecasting of college football games. This role expansion was initiated by some members concerns that the increase in telecasting of college football games would reduce public attendance at both those and non-televised games. The NCAA, by its television restrictions, became the sole authority empowered to negotiate the assignment of television broadcasting rights of collegiate football games. Individual schools were prohibited from negotiating with television or cable outlets. The enacted television controls limit both the number of times any one school may appear on television, and the amount of revenue any one school may receive from television appearances. The NCAA asserts that its television restrictions promote competitive balance and enhances interest in all college football games.

The remaining portion of this section chronologically traces the events and issues that led directly to the plaintiffs' action to initiate the lawsuit against the NCAA.

In 1976, sixty major football colleges, all members of the NCAA, discontented with different aspects of the NCAA regulations governing college football, banded together and formed the Collegiate Football

Association (CFA). Membership in the CFA is restricted to schools meeting certain criteria of size and importance with respect to their football programs. The original purpose of the CFA was to lobby and promote the interests of the prominent football schools within the NCAA. Eventually, however, members of the CFA felt that those colleges with major football programs should have a greater voice in the formulation of football television policy within the NCAA.

In December 1980, the CFA began investigating the possibility of negotiating an agreement of its own with the major television networks for the broadcast rights to games involving CFA member teams. To support this action, the members of the CFA adopted the position that nothing in the NCAA Constitution or Bylaws empowered the NCAA to act as the exclusive bargaining agent on behalf of all its members for the sale of television rights to college football games.

In response to the CFA's position, the NCAA Council adopted an "Official Interpretation" of the Bylaws which held that the NCAA controls all forms of televising intercollegiate football games of its member institutions. In addition, the interpretation stated that any commitment by a member institution with respect to televising of its football games in future seasons is subject to the terms of the NCAA Football Television Plan applicable to such season.

In July 1981, the NCAA announced a four-year, $263.5 million contract with ABC Sports, Inc., and CBS Sports, Inc., and a two-year $17 million contract with Turner Broadcasting Systems (TBS), Inc. The contract featured a doubling of revenues over the existing ABC contract, and allowed for an increase in the number of appearance

opportunities for any one member school. In August 1981, the CFA announced a four-year agreement with NBC Sports, Inc., for $180 million, pending final approval by the CFA member schools. In late August, the agreement was narrowly ratified by a non-binding vote of the CFA membership.

In response, high ranking officials of the NCAA made it evident that CFA members, if they chose to be bound by the NBC contract, would be in violation of NCAA rules, and such infraction was punishable by means (ranging from reprimands to explusion) provided by NCAA legislation.

In September 1981, the Universities of Oklahoma and Georgia filed a class action suit against the NCAA in the U.S. District Court of the Western District of Oklahoma, in behalf of all CFA members (many of whom later denounced the action and subsequently refused to finance it). The plaintiffs alleged that the controls enunciated in the "1982-85 NCAA Television Plan" violate the Sherman Antitrust Act, 15 U.S.C. §1-2 (1980). The plaintiffs asserted that to facilitate the NCAA arrangement, individual schools are precluded from exercising independent judgment with regard to output and pricing decisions. The CFA was granted a temporary injunction prohibiting the NCAA from attempting to interfere with the CFA's negotiations with NBC. Nevertheless, the CFA failed to consummate its agreement with NBC because it was unable to provide an adequate inventory of teams.

In October 1981, the NCAA Council, at the request of the members of the CFA, voted to convene a special convention in December 1981, to consider the restructuring of the NCAA. As mentioned earlier,

the NCAA member schools are divided into several divisions according to the size of the institution and its intercollegiate athletic program. Division I consisted of about 275 institutions with major athletic programs. Less than 190 of these institutions field intercollegiate varsity football programs. Divisions II and III included approximately 500 institutions with less extensive athletic programs. For football purposes, Division I was split into two subdivisions, I-A and I-AA. Members of the CFA and other prominent football schools belong to Division I-A. The CFA members held that their interest and those of schools with relatively minor football programs (also belonging to Division I-A) were not consistent, and that Division I should be restructured to reduce the number of Division I-A schools. In December 1981, the NCAA membership at the special convention, adopted a reorganization plan that reduced the 137 schools in Division I-A to 96 for football purposes only.

At its annual convention, held in January 1982, the NCAA general membership overwhelmingly voted to solidify the NCAA's control of the football television rights of its membership. They adopted an amendment to the NCAA Bylaws which established the propriety of the NCAA television controls, and ratified the NCAA Council's "Official Interpretation" (mentioned earlier). The CFA members vehemently objected to the fact that all NCAA members, whether or not they have intercollegiate football programs, were allowed an equal vote in the adoption of the amendment and the ratification of other football television policies.

In June 1982, United States District Court Judge Juan C. Burciaga of New Mexico opined as follows:

> (1) The television controls exercised by the NCAA constitute a horizontal agreement among competitors to fix prices and restrict output, in violation of 15 U.S.C. §1; and
>
> (2) The NCAA exercises monopoly power over the market of collegiate football television, in violation of 15 U.S.C. §2.[1]

This decision upheld the plaintiffs' position that schools own the television rights to their football games and that the NCAA cannot require a school to assign such rights as an obligation of membership.

Many prominent members of the NCAA, dismayed over the court's decision, expressed feelings of undertainty, doubt, and confusion regarding the effects of a competitive free market on collegiate football. Traditionalists held the viewpoint that competition amongst the nation's colleges for television revenues is an unwise policy objective. They surmised that "a mad scramble for TV revenues . . . would make rich schools richer and drive their less glamorous rivals to gridiron extinction," and that the court's decision "could break up conferences because stronger schools, reluctant to share their new-found television riches with other members, would be tempted to go it alone."[2]

It is not surprising that apprehension surrounded the court's decision. The intercollegiate football television market had been under NCAA control since the early 1950s. The only recent public information concerning viable alternatives to strict NCAA control became available when members of the CFA negotiated a contract with NBC Sports, Inc. This exclusive contract was economically more

rewarding than the NCAA contracts with ABC and CBS, and this development may have possibly suggested to many that competition amongst the networks could result in even more lucrative future arrangements. Despite its appearance as competitive, the exercise reveals little information about the price-determining mechanisms of a competitive television market. It is questionable whether or not free-market negotiations would establish what the court termed "supracompetitive" levels of revenue. This is especially troublesome in the light of the recent decline in college football television ratings, reflecting the general trend in TV sports. In addition, the District Court's opinion suggests that future competitive free market revenues for regionally and nationally televised games will perforce differ from those established under the NCAA plan. Judge Burciaga stated:

> The evidence suggests the fact that the networks are actually paying the large fees because the NCAA agrees to limit production. If the NCAA would not agree to limit production, the networks would not pay so large a fee. Because the NCAA limits production, the networks have not feared their broadcasting will have to compete head-to-head with other college football telecast, either on the other networks or on various local stations.[3]

Another consequence of the court's decision will be the proliferation of the number of televised games on local and regional levels. In his opinion, Judge Burciaga stated:

> The evidence is clear that many television broadcasters, both national networks and local stations, would buy the right to televise many individual college football games. Indeed, the evidence shows that many more games would be televised in a free market than televised under NCAA controls. It is likely that fewer schools would appear on network broadcasts. The testimony of network executives makes clear that if it were up to them, they would show fewer teams than they are required to show under NCAA controls. At the same time, however, the evidence shows that many more teams would be shown on local and regional broadcasts.[4]

The NCAA, following the adverse decision, obtained an injunction pending an appeal to the 10th United States Circuit Court of Appeals in Denver, Colorado. In May 1983, the 10th U.S. Circuit Court of Appeals, in a two to one decision, upheld the District Court's ruling. The NCAA then announced that it would seek to appeal the ruling to the United States Supreme Court. In June 1983, Supreme Court Justice Byron White granted the NCAA a stay of the 10th Circuit Court decision pending the granting for review by the Supreme Court. This action extended the ABC-CBS-TBS contracts through the 1983 season. In December 1983, the Supreme Court announced that it would review the case. In March 1984, oral arguments were presented to the Supreme Court. Finally, in June 1984, the Supreme Court, in a seven to two decision, upheld the District Court's decision, and ruled that the NCAA Television Plan for the 1982-85 seasons to be in violation of federal antitrust law. Consequently, all members of the NCAA are free to independently negotiate and contract for the sale of television broadcasting rights to their universities' football games.

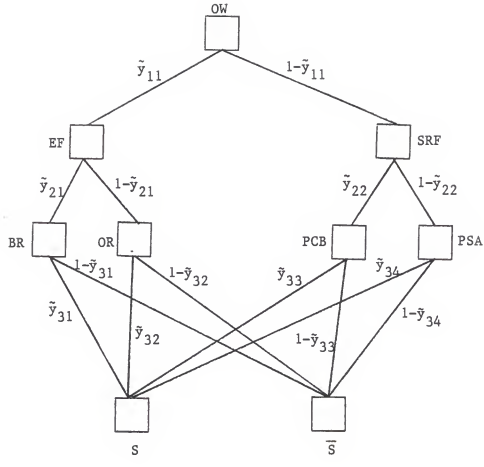
Decision Problem Structure and Modeling

The previous section pointed out several economic and non-economic (i.e., sports-related, organizational, and political) factors that appear to weigh heavily upon the plaintiffs' decision to file suit. Unfortunately, the exact nature and identity of the most important decision factors, and the extent to which they influence the final decision, are not publicly known. This is also the case for those decision agents that chose not to file suit against the NCAA. This

situation is in part attributable to the fact that athletic directors tend not to publicly discuss the logic of why and how they reach their decisions. The closed and highly political nature of conferences and similar affiliations further exacerbates this situation.

Given the above situation, the purpose of this section is to hierarchically structure the decision problem facing the decision agents of the football colleges of the NCAA. This task is performed from the perspective of the author as an investigative agent with incomplete information under uncertainty. Since the process of modeling and analyzing decision problems in this fashion is removed from reality, the decision models should be kept as streamlined as possible. This is desirable both from a philosophical standpoint and from a practical standpoint, since the overall effect is to minimize the labor involved in conducting the probabilistic analysis, and to ease the interpretation of the derived results. With this in mind, the decision problem is hierarchically structured as shown in Figure 6.1. The remaining portion of this section discusses why the decision model is so structured and organized.

It is reasonable to assume that the decision to file suit rests upon whether or not the consequences of such an action will be economically beneficial to the deciding agent. This is a prickly issue, since the economic effects of a competitive television market in collegiate football are uncertain, as are the possible alternatives in a modified NCAA-controlled television market. In addition, the decision agent must consider whether or not collegiate football (or collegiate sports in general) will benefit from such an action.



OW - Overall Welfare of the Athletic Program
EF - Economic Factors
SRF - Sports-Related Factors
BR - Broadcast Revenues
OR - Other Revenues
PCB - Promotion of Competitive Balance
PSA - Preservation of the Student Athlete
S - File Suit
$\bar{S}$ - Do Not File Suit

Figure 6.1. A Hierarchically Structured Model of the Decision of Whether to File Suit.

These aspects of the decision problem are incorporated into the decision model of Figure 6.1. Deciding which action to take (within the context of the AHP paradigm) requires the decision agent to assess the relative importance of each of the economic and sports-related factors which impact on the overall welfare of the athletic program. The decision agent must also assess the impact of each of the alternative actions upon these factors, and then decide which action most favorably influences the welfare of the athletic program.

The economic considerations impacting the welfare of a specific athletic program are influenced by various sources of revenue. Broadcast revenues are considered to be those revenues associated with national, regional, or local telecasting of college football contests. If the athletic program of a decision agent is affiliated with a conference, broadcast revenues from national or regional contests of all conference members may be pooled, and then allocated to each member according to an agreed upon sharing rule. Television revenues earned by independent football schools are not shared with other schools. Athletic programs also receive other revenues from alumni contributions to the football program, and from a percentage of total gate receipts from both home and away games. Consequently, the action to file suit must be weighed in terms of the comparative impacts on all sources of revenue, and their subsequent impact on the welfare of the athletic program.

The sports-related considerations impacting the welfare of an athletic program are difficult to specify and measure, because the major issues underlying the decision agent's action are economic in

nature. Assuredly, the effects of either course of action will profoundly impact on the organizational roles, structure, influence, and power of and benefits of membership in the NCAA, CFA, athletic conferences, and similar affiliations. As shown in Figure 6.1, the sports-related considerations are influenced by issues regarding the promotion of competitive balance and the protection of amateurism in collegiate sports. Athletically balanced competition is formally promoted by the NCAA via rules which regulate recruiting, set requirements for and the number of scholarships that may be offered by any one school, fix the number of coaches a team may have, etc. Similarly, scholarship and amateurism are formally promoted by rules which set minimum academic and eligibility requirements consistent with the notion of the "scholar athlete." There is, however, a commercial side to the issues of competitive balance and amateurism. Specifically, most sources of revenue are influenced in part by the competitive quality of teams within a conference (or a similar affiliation) as well as the relative competitive quality of the conference with respect to other conferences. In turn, competitive quality is affected by the quality of the athletes the conferences are able to attract and recruit, which in turn depends upon the economic vitality of the conference teams, etc. Thus, any change in the economic environment of a conference that affects the distribution of football power or affects the closeness of conference title races (i.e., the competitive balance of the conference teams, or conferences themselves) is of great concern to the decision agent. Especially when the decision agent can profoundly impact this environment by a successful law suit

against the NCAA. Similarly, an increase in professionalism of college football can call into question the role of football as an integral part of college life within institutions of higher education, as well as question the efficacy of rules which protect amateurism and scholasticism in collegiate sports. The effectiveness of the NCAA as a rule-making body in these traditional sports areas depends upon its power to punish violators. Thus, any change or reduction (directly or indirectly) in the rule-enforcing potency (specifically, the power to banish violators from television appearances) may be of special interest to decision agents weighing the consequences of the various alternative actions.

Probabilistic Analysis of the Decision to Sue or Not to Sue

The purpose of this section is to conduct a probabilistic analysis of the assignment of priorities in the hierarchically structured decision problem of Figure 6.1. The goal is to mathematically model the choice problem in terms of the parameters characterizing the decision agent's relative importance judgments. The object is to determine what relationships between, and the values (or range of values) of, the various parameters must be in order to substantiate the observed choice or outcome. In this way, a range of relative importance judgments can be investigated and interpreted with the purpose of gaining insight into the decision-making processes of the agents involved.

The random variables $\tilde{y}_{ij}$ of Figure 6.1 represent the priority weights assigned independently to the j^{th} decision element of the i^{th} level of the hierarchy. For purposes of analysis assume that:

$$\tilde{y}_{11} \sim \beta(\alpha_1,\alpha_2) \tag{6.1a}$$

$$\tilde{y}_{21} \sim \beta(\gamma_1,\gamma_2) \tag{6.1b}$$

$$\tilde{y}_{22} \sim \beta(\zeta_1,\zeta_2) \tag{6.1c}$$

$$\tilde{y}_{31} \sim \beta(\eta_1,\eta_2) \tag{6.1d}$$

$$\tilde{y}_{32} \sim \beta(\eta_3,\eta_4) \tag{6.1e}$$

$$\tilde{y}_{33} \sim \beta(\eta_5,\eta_6) \tag{6.1f}$$

and

$$\tilde{y}_{34} \sim \beta(\eta_7,\eta_8). \tag{6.1g}$$

Consider the transformation

$$T: \quad \tilde{t}_1 = \tilde{y}_{11}\tilde{y}_{21}\tilde{y}_{31} \tag{6.2a}$$

$$\tilde{t}_2 = \tilde{y}_{11}\tilde{y}_{21}(1-\tilde{y}_{31}) \tag{6.2b}$$

$$\tilde{t}_3 = \tilde{y}_{11}(1-\tilde{y}_{21})\tilde{y}_{32} \tag{6.2c}$$

$$\tilde{t}_4 = \tilde{y}_{11}(1-\tilde{y}_{21})(1-\tilde{y}_{32}) \tag{6.2d}$$

$$\tilde{t}_5 = (1-\tilde{y}_{11})(\tilde{y}_{22})(\tilde{y}_{33}) \tag{6.2e}$$

$$\tilde{t}_6 = (1-\tilde{y}_{11})(\tilde{y}_{22})(1-\tilde{y}_{33}) \tag{6.2f}$$

$$\tilde{t}_7 = (1-\tilde{y}_{11})(1-\tilde{y}_{22})(\tilde{y}_{34}) \tag{6.2g}$$

$$\tilde{t}_8 = (1-\tilde{y}_{11})(1-\tilde{y}_{22})(1-\tilde{y}_{34}). \tag{6.2h}$$

The inverse transformation is

$$T^{-1}: \quad \tilde{y}_{11} = (\tilde{t}_1+\tilde{t}_2+\tilde{t}_3+\tilde{t}_4) = \tilde{t}_{1\to4} \tag{6.3a}$$

$$\tilde{y}_{21} = \frac{(\tilde{t}_1 + \tilde{t}_2)}{\tilde{t}_{1\to4}} \tag{6.3b}$$

$$\tilde{y}_{22} = \frac{(\tilde{t}_5 + \tilde{t}_6)}{(1 - \tilde{t}_{1\to4})} \tag{6.3c}$$

$$\tilde{y}_{31} = \frac{\tilde{t}_1}{(\tilde{t}_1 + \tilde{t}_2)} \tag{6.3d}$$

$$\tilde{y}_{32} = \frac{\tilde{t}_3}{(\tilde{t}_3 + \tilde{t}_4)} \tag{6.3e}$$

$$\tilde{y}_{33} = \frac{\tilde{t}_5}{(\tilde{t}_5 + \tilde{t}_6)} \tag{6.3f}$$

$$\tilde{y}_{34} = \frac{\tilde{t}_7}{1-(\tilde{t}_1+\tilde{t}_2+\tilde{t}_3+\tilde{t}_4+\tilde{t}_5+\tilde{t}_6)} = \frac{\tilde{t}_7}{1 - \tilde{t}_{1\to6}}. \tag{6.3g}$$

The absolute value of the Jacobian of the transformation (6.2) is

$$|J| = [(\tilde{t}_1+\tilde{t}_2)(\tilde{t}_3+\tilde{t}_4)(\tilde{t}_5+\tilde{t}_6)(\tilde{t}_{1\to4})(1-\tilde{t}_{1\to4})(1-\tilde{t}_{1\to6})]^{-1}. \tag{6.4}$$

The j.p.d.f. of the vector random variable $\tilde{y} = (\tilde{y}_{11},\tilde{y}_{21},\tilde{y}_{22},\tilde{y}_{31},\tilde{y}_{32},\tilde{y}_{33},\tilde{y}_{34})$ can be uniquely determined by the product of its m.p.d.f.'s; that is,

$$f_y(y_{11},y_{21},y_{22},y_{31},y_{32},y_{33},y_{34}) =$$

$$K\, y_{11}^{\alpha_1-1}(1-y_{11})^{\alpha_2-1}\, y_{21}^{\gamma_1-1}(1-y_{21})^{\gamma_2-1}\, y_{22}^{\zeta_1-1}(1-y_{22})^{\zeta_2-1}$$

$$\cdot y_{31}^{\eta_1-1}(1-y_{31})^{\eta_2-1}\, y_{32}^{\eta_3-1}(1-y_{32})^{\eta_4-1}\, y_{33}^{\eta_5-1}(1-y_{33})^{\eta_6-1}\, y_{34}^{\eta_7-1}(1-y_{34})^{\eta_8-1}$$

$$\text{for } 0 < \tilde{y}_{ij} \le 1, \quad i=1,j=1;\ i=2,j-1,2;\ i=3,j-1,2,3,4 \qquad (6.5)$$

where

$$K = [B(\alpha_1,\alpha_2)B(\gamma_1,\gamma_2)B(\zeta_1,\zeta_2)B(\eta_1,\eta_2)$$

$$\cdot\, B(\eta_3,\eta_4)B(\eta_5,\eta_6)B(\eta_7,\eta_8)]^{-1}.$$

The j.p.d.f. of the vector random variable $\tilde{t} = (\tilde{t}_1,\ldots,\tilde{t}_7)$ is obtained by applying the transformation (6.2) (that is, substituting equations (6.3) into density (6.5) and multiplying the result by (6.4)). Thus, after rearranging,

$$g_t(t_1,\ldots,t_7) = K \prod_{i=1}^{7} t_i^{\eta_i-1}\,(1-\sum_{i=1}^{7} t_i)^{\eta_8-1}$$

$$\cdot\,(t_1+t_2)^{\gamma_1-(\eta_1+\eta_2)}(t_3+t_4)^{\gamma_2-(\eta_3+\eta_4)}(t_5+t_6)^{\zeta_1-(\eta_5+\eta_6)}$$

$$\cdot\,(t_{1\to4})^{\alpha_1-(\gamma_1+\gamma_2)}(1-t_{1\to4})^{\alpha_2-(\zeta_1+\zeta_2)}$$

$$\cdot\,(1-t_{1\to6})^{\zeta_2-(\eta_7+\eta_8)} \qquad (6.6)$$

for $\tilde{t}_i > 0$, $i = 1,\ldots,7$, and $\sum_{i=1}^{7} \tilde{t}_i \le 1$, and zero elsewhere.

As in the previous chapter, assume that

$$\alpha_1 = \gamma_1 + \gamma_2, \tag{6.7a}$$

$$\alpha_2 = \zeta_1 + \zeta_2, \tag{6.7b}$$

$$\gamma_1 = \eta_1 + \eta_2, \tag{6.7c}$$

$$\gamma_2 = \eta_3 + \eta_4, \tag{6.7d}$$

$$\zeta_1 = \eta_5 + \eta_6, \tag{6.7e}$$

and

$$\zeta_2 = \eta_7 + \eta_8. \tag{6.7f}$$

Substitution of equations (6.7) into density (6.6) yields

$$g_t(t_1,\ldots,t_7) = \frac{\Gamma(\sum_{i=1}^{7} \eta_i)}{\prod_{i=1}^{7} \Gamma(\eta_i)} \prod_{j=1}^{7} t_j^{\eta_j - 1} \left(1 - \sum_{i=1}^{7} t_i\right)^{\eta_8 - 1} \tag{6.8}$$

at any point in the simplex: $S_7:\{(t_1,\ldots,t_7): t_i > 0$ for $i = 1,\ldots,7$, and $\sum_{i=1}^{7} t_i \leq 1\}$, in R_7 and zero outside, where η_i are all real and positive. Thus, the vector random variable

$$\underset{\sim}{t} = (t_1,\ldots,t_7) \sim D(\eta_1,\ldots,\eta_7;\ \eta_8).$$

The total priority weight assigned to the decision to file suit (S) is the sum of the priority weights $t_1 + t_3 + t_5 + t_7$. With this in mind, define

$$z = t_1 + t_3 + t_5 + t_7 \tag{6.9a}$$

and

$$1 - \tilde{z} = \tilde{t}_2 + \tilde{t}_4 + \tilde{t}_6 + \tilde{t}_8. \tag{6.9b}$$

Then, the probability that the option to sue is assigned more priority weight than the option of not sueing can be written as

$$P[S > \overline{S}] = P[\tilde{z} > 1 - \tilde{z}]. \tag{6.10}$$

The distribution over $\tilde{z}$ can be found using Property 3.2, that is

$$\tilde{z} \sim \beta(\eta_1+\eta_3+\eta_5+\eta_7,\ \eta_2+\eta_4+\eta_6+\eta_8). \tag{6.11}$$

Hence, equation (6.10) can be written as

$$P[S > \overline{S}] = P[\tilde{z} > 1/2] = 1 - \int_0^{1/2} \frac{z^{\alpha-1}(1-z)^{\beta-1}}{B(\alpha,\beta)}\, dz \tag{6.12}$$

where $\alpha = \eta_1+\eta_3+\eta_5+\eta_7$ and $\beta = \eta_2+\eta_4+\eta_6+\eta_8$.

Equation (6.12) can also be written as

$$P[S > \overline{S}] = P[\tilde{z} > 1/2] = 1 - \sum_{j=\alpha}^{\alpha+\beta-1} \binom{\alpha+\beta-1}{j} \left(\frac{1}{2}\right)^{\alpha+\beta-1} \tag{6.13}$$

provided that β is integral (see Raiffa and Schlaifer, 1962, p. 218).

Equations (6.12) or (6.13) with equation (6.11) provides a mathematical model of the decision agent's choice model, and the parameters η_i can be varied to reflect perceived importance judgments attributed to the decision agent under investigation.

In a nutshell, the analysis of the decision agent's problem can be stated as follows: Given that $\tilde{z} \sim \beta(\eta_1+\eta_3+\eta_5+\eta_7,\ \eta_2+\eta_4+\eta_6+\eta_8)$, what

must be the values (or the range of values) of the parameters η_i necessary to satisfy the probability $P[S > \overline{S}] = P[\tilde{z} > 1/2](\leq,=,\geq)k$, subject to a choice of k ($0 \leq k \leq 1$) and one of the relations $(\leq,=,\geq)$? The analysis of this problem is facilitated by noting that if $k = 0.50$, we have

$$\eta_1+\eta_3+\eta_5+\eta_7 = \eta_2+\eta_4+\eta_6+\eta_8. \tag{6.14}$$

This relationship follows from the symmetry property of the beta distribution about 1/2 if the parameters characterizing the distribution are equal to one another. Consequently, the probability that $\tilde{z}$ is greater than $1 - \tilde{z}$ must be equal to 0.50. Within the context of the current problem, if equality of (6.14) holds, then the decision agent is indifferent between the two actions S and $\overline{S}$ (in terms of their relative relation or importance to the objective OW). If the equality of (6.14) does not hold, then the priority distribution over $\tilde{z}$ becomes skewed towards the right (left) if the right hand side of (6.14) increases relative to the left hand side. This indicates that the action to file suit (not to file suit) receives more (less) priority weight relative to its impact upon the overall welfare of the athletic program (OW).

Equation (6.14) can be rearranged and rewritten as follows:

$$(\eta_1-\eta_2) + (\eta_3-\eta_4)(\leq,=,\geq)(\eta_6-\eta_5) + (\eta_8-\eta_7) \tag{6.15}$$

where the defining relation $(\leq,=,\geq)$ is selected to indicate relative preference or indifference among the actions S and $\overline{S}$ relative to OW, as described above in equation (6.14).

The next section explores several combinations of importance judgments over the decision problem elements with respect to possible outcomes in a probabilistic framework. The object is to gain some insight into the preferences underlying the decision agent's choices.

Interpretation

The difference terms of equation (6.15) reflect the decision agent's perceptions regarding the relative importance of either action (S or $\overline{S}$) upon broadcast and other revenues, and upon the promotion of competitive balance and the protection of amateurism in college football. The sign and magnitude of these terms indicate the strength of importance of each action upon these relevant factors. This section considers the contribution of each of these difference terms to the plaintiffs' decision to file suit. In addition, these terms are also discussed in a variety of other preference contexts that reflect the general views and concerns held by nonlitigious decision agents. The upshot is to gain insight into the decision-making processes underlying the actions taken by these decision agents.

The plaintiffs' decision to file suit was primarily motivated by economic interests; this position is clearly stated in Judge Burciaga's opinion:

> Oklahoma's intercollegiate football program has, over the years, produced many outstanding and highly-ranked teams. Oklahoma is capable of attracting large national television audiences for its televised games. Football is the only sport sponsored by Oklahoma which actually generates revenue beyond the costs of fielding a team. The profits generated by the football team support all of the other sixteen

> men's and women's sports in which Oklahoma participates. In recent years, these programs have become more and more expensive to maintain due to the increasing costs and the expansion of athletic programs for women. As a result, Oklahoma seeks to maximize its revenues from football television. Like Oklahoma, Georgia has compiled a record of outstanding success with its intercollegiate football program, and football is the major revenue producing sport for Georgia. Budgetary pressures have forced cuts in Georgia's athletic program, including the elimination of its intercollegiate wrestling program. Georgia also seeks to maximize the revenues generated from the televising of its football games.[5]

This position clearly indicates the prominent influence of the economic factors upon the overall welfare of the plaintiffs' athletic programs. Hence, we can infer that $\alpha_1 >> \alpha_2$; that is

$$\eta_1+\eta_2+\eta_3+\eta_4 >> \eta_5+\eta_6+\eta_7+\eta_8. \tag{6.16}$$

The plaintiffs' position regarding the importance of maximizing television broadcasting revenues also suggests that $\gamma_1 > \gamma_2$, $\gamma_1 > \zeta_1$, and $\gamma_1 > \zeta_2$; that is

$$\eta_1 + \eta_2 > \eta_3 + \eta_4 \tag{6.17a}$$

$$\eta_1 + \eta_2 > \eta_5 + \eta_6 \tag{6.17b}$$

$$\eta_1 + \eta_2 > \eta_7 + \eta_8, \tag{6.17c}$$

respectively.

The plaintiffs' belief that broadcast revenues are significantly more likely to be maximized in a competitive collegiate football television market is reflected in Judge Burciaga's opinion:

> There can be no doubt that both of these plaintiffs would have realized increased revenues from football television but for the NCAA controls. The fact that

> ABC paid the same fee for the plaintiffs' games as it did for the less attractive games leads readily to the inference that if the networks were allowed to choose games for broadcast free from the strictures of the NCAA controls, schools such as the plaintiffs' would receive larger fees for their games. This is particularly true if the networks are bidding against one another for the right to broadcast any particular game. It is also clear that the plaintiffs would realize more revenue from local and regional broadcasts were it not for the NCAA controls.[6]

Thus, we can infer that $\eta_1 >> \eta_2$. Hence, equations (6.15) and (6.17) can be written as follows:

$$\eta_1 + (\eta_3 - \eta_4)R(\eta_6 - \eta_5) + (\eta_8 - \eta_7) \tag{6.18}$$

and

$$\eta_1 > \eta_3 + \eta_4 \tag{6.19a}$$

$$\eta_1 > \eta_5 + \eta_6 \tag{6.19b}$$

$$\eta_1 > \eta_7 + \eta_8, \tag{6.19c}$$

respectively, where R is the defining relation selected appropriately from the set $\{<,=,>\}$. Since the inequalities of equations (6.19) hold, it immediately follows that

$$\eta_1 > \eta_3 - \eta_4 \tag{6.20a}$$

$$\eta_1 > \eta_6 - \eta_5 \tag{6.20b}$$

and

$$\eta_1 > \eta_8 - \eta_7. \tag{6.20c}$$

The selection of R from $\{<,=,>\}$ depends upon the difference terms on the right hand side of equations (6.20).

The relationship between parameters η_3 and η_4 can be generally inferred from a statement found in Judge Burciaga's opinion:

> Dr.s William Banowsky and Fred Davison, Presidents of [the Universities of] Oklahoma and Georgia, respectively, made clear that the antitrust activities of the NCAA do not destroy the profitability of their football programs. The plaintiffs will continue to play football whatever the ultimate outcome of litigation, and they will do so at a profit.[7]

This statement, coupled with the plaintiffs' ascertion that there is no evidence supporting the claim that NCAA television restrictions protect live attendance at football contests, implies that $\eta_3 \cong \eta_4$. That is, this suggests that the plaintiffs are indifferent to the relative impact of either collegiate football television scenario on other revenue sources (that is, gate receipts, alumni contributions, etc.).

The relationship between the parameters η_5 and η_6 can be generally inferred from the plaintiffs' position regarding the issue of the creation of a "power elite." The NCAA, according to Judge Burciaga's opinion, offered the following argument in defense of its television control policies:

> The increased revenues which schools such as Georgia and Oklahoma would receive in an open market would give them an overwhelming advantage over other schools. The large revenues could be channeled into recruitment and training of better and better football teams. Schools which did not generate such revenues could never . . . hope to compete with the "power elite" of fifteen or twenty schools which would result from a free market situation.[8]

The plaintiffs argued that despite the NCAA's regulatory role in collegiate television there exists substantive evidence supporting the existence of such a "power elite." The plaintiffs argued that it is not an impossibility for schools with lesser-known football programs

to achieve dramatic increases in national reputation and recognition, with a concomitant increase in television exposure and revenue. From these arguments it can be inferred that the plaintiffs held that threats to competitive balance, at least from the standpoint of the creation of imbalances in the distribution of the playing quality among football programs (at least those of the Division I-A or CFA member schools), are equally likely to occur in either college football television market scenario. This reasoning implies that $\eta_5 \cong \eta_6$.

Given the above discussion, equation (6.18) can be written as

$$\eta_1 \; R \; (\eta_8 - \eta_7), \tag{6.21a}$$

however, equation (6.20c) forces the relationship

$$\eta_1 > (\eta_8 - \eta_7). \tag{6.21b}$$

Equation (6.21b) follows from the assumption that broadcast revenues are operationally more important than sports-related factors to the overall welfare of the plaintiffs' football programs.

It is of interest to investigate preference perspectives other than those held by the plaintiffs; specifically those of a nonlitigious member of the CFA. As in the case of the plaintiffs' decision, this investigation utilizes an interpretive analysis of the difference terms of equation (6.15); that is,

$$(\eta_1 - \eta_2) + (\eta_3 - \eta_4) \; R \; (\eta_6 - \eta_5) + (\eta_8 - \eta_7). \tag{6.22}$$

To simplify analysis, assume that $\eta_8 - \eta_7 = 0$. This relationship assumes that the decision agent is indifferent between broadcasting

scenarios relative to their impact upon the promotion and protection of amateurism in collegiate athletics. This would follow if the decision agent believed that the NCAA would continue to "effectively" guard amateurism regardless of the outcome of the decision to file suit. Thus, equation (6.22) becomes

$$(\eta_1 - \eta_2) + (\eta_3 - \eta_4)\ R\ (\eta_6 - \eta_5) \tag{6.24}$$

where R is the defining relation selected from $\{\leq,=,\geq\}$ depending upon the sign and magnitude of the difference terms in combination.

In the interpretative analysis of the plaintiffs' decision, the term $\eta_6 - \eta_5$ was assumed to be zero. There exists, however, decision agents who assert that $\eta_6 - \eta_5 > 0$. This relation reflects the concerns about the creation of an economic "power elite" as a consequence of a competitive television market, as well as concerns regarding the NCAA's ability to function effectively as a rule-enforcer in a competitive market.

The analysis of the plaintiffs' decision also assumed that the term $\eta_3 - \eta_4 \cong 0$. The possibility does exist that $\eta_3 - \eta_4 > 0$, or $\eta_3 - \eta_4 < 0$. A competitive market, depending upon the circumstances of an individual decision agent's athletic program, can affect live attendance as argued by the NCAA. It is not clear whether alumni contributions would decline if live attendance decreased, or that increased television exposure would induce greater alumni interest. With the possibility of offsetting effects, or at what appears to be marginal increases or decreases in other revenues, it seems reasonable that the assumption that $\eta_3 \cong \eta_4$ is justifiable for the majority of CFA member schools.

Given the above arguments, equation (6.24) becomes

$$(\eta_1 - \eta_2) \; R \; (\eta_6 - \eta_5) > 0. \tag{6.25}$$

Thus, the decision to file suit rests on whether the decision agent believes that the economic benefits of a competitive college football television market offset the economic costs of competitive imbalances threatened by the revocation of the NCAA television controls.

If $\eta_1 >> \eta_2$, and $\eta_1 > \eta_5 + \eta_6$, the decision to file suit governed by equation (6.25) follows that of the plaintiffs'. Let $|(\eta_1 - \eta_2) - (\eta_6 - \eta_5)| \leq c > 0$, define an indifference zone. That is, in this zone the decision agent is indifferent to the relative impact of either television scenario upon the welfare of the athletic program. Thus, it seems unlikely that an indifferent decision agent in this zone would file suit (even if $\eta_1 > \eta_2$) unless there were mitigating factors operating which are not captured in the decision model of Figure 6.1. If this is the case, then the decision agent would not file suit if $\eta_1 < \eta_2$, rather than requiring $\eta_1 << \eta_2$. That is, the decision agent need only consider the impact of a competitive television market to be detrimental to the athletic program; the degree of detriment need not be qualified to support the decision. On the other hand, the indifference zone conceptualization of the problem suggests that $\eta_1 >> \eta_2$ before the action to file suit is supported.

The decision not to file suit (i.e., $\eta_1 < \eta_2$) is supported by the concern of many decision agents that an increase in television exposure in a competitive market does not necessarily lead to increased television revenues. It is feared that a glut of televised

college football contests will depress revenues for all games across the board, except for a few "blue chip" games played by the "power elite" of collegiate football.

Conclusion

It is unrealistic to expect that decision agents in collegiate athletics employ formal decision-making models and procedures, or that they should do so. It is realistic to expect that expert decision agents in collegiate athletics arrive at many of their decisions through an implicit decision-making process. Through an analysis of the previous decision problem using a formal probabilistic framework from the standpoint of an investigative agent, some insight was gained about the implicit decision-making process.

The formal analysis of the Universities of Oklahoma and Georgia's decision to file suit against the NCAA has revealed a singular lack of appreciation of the potential consequences (both direct or indirect) of each party's action. A comparison of the economic and sports related consequences of either collegiate football television scenario suggests that perhaps some alternative nonlegal action might have produced results beneficial to both parties. The potentially serious economic effects of a competitive television market upon the viability of Division I-AA, Division II and III athletic programs certainly presents an interesting basis for cooperation and compromise.

The initially observable consequences of the suit action are not without an element of irony. For example, most of the decision agents representing the prominent college football schools (which includes

the plaintiffs) have recognized the need for some kind of voluntary output control or restrictions which would avoid a situation in which individual schools would negotiate separately for the assignment of its broadcasting rights. Voluntary packages of this kind (i.e., either conference packages, or CFA agreements, etc.), however, may be subject to antitrust scrutiny under the Supreme Court's ruling. Also, the plaintiffs' implicit assumption that each will be economically better off in a competitive television market is in doubt. Specifically, "when the Sooners (i.e., the University of Oklahoma) put out their (football) games for TV bids twice this spring (1984), they pulled them back both times because the offers were too low."[9] This suggests that even among the plaintiffs, the differential effects of specific geographic and television broadcasting environments in a competitive television market were not clearly comprehended.

The subject of the effects a competitive college football television market on collegiate athletics is not completely closed. The House Subcommittee on Oversight and Investigations of the United States Congress plans to hold hearings on July 31, 1984, on this subject. According to "Representative Edward J. Markey, Democrat of Massachusetts and a member of this subcommittee. . . . 'The purpose of this hearing is to examine the response of broadcasting to the dramatic Supreme Court decision, and also determine the impact on colleges and universities at all levels.'"[10] The results of such an investigation, if significant, may present a subject worthy of continued research and investigation in this area.

Notes

[1]See Board of Regents of the University of Oklahoma, and the University of Georgia Athletic Association v. National Collegiate Athletic Association, Civil No. 81-1209-BV (Western District of Oklahoma, 1982), p. 1. Hereafter referred to as Board of Regents of the University of Oklahoma et al. v. NCAA.

[2]See Kirschenbaum (1982), p. 9.

[3]See Board of Regents of the University of Oklahoma et al. v. NCAA, p. 27.

[4]See Board of Regents of the University of Oklahoma et al. v. NCAA, p. 53.

[5]See Board of Regents of the University of Oklahoma et al. v. NCAA, p. 3.

[6]See Board of Regents of the University of Oklahoma et al. v. NCAA, p. 42.

[7]See Board of Regents of the University of Oklahoma et al. v. NCAA, p. 60.

[8]See Board of Regents of the University of Oklahoma et al. v. NCAA, p. 59.

[9]See Creamer (1984), p. 9.

[10]See White (1984), p. 20.

APPENDIX A
MATHEMATICAL APPENDIX TO CHAPTER III

This appendix provides a proof (an n-variate generalization motivated by Kotlarski's (1967) work) of Theorem 3.1. The proof is straightforward once an important underlying lemma is made evident. The first part of this appendix formally states and proves the essential lemma, whereafter the theorem is proved.

Lemma (Kotlarski, 1967). Let $\tilde{v}_1,\ldots,\tilde{v}_n$ be independent real random variables, and let

$$\tilde{x}_{ij} = \tilde{v}_i/\tilde{v}_j, \qquad i,j = 1,\ldots,n,\ i \neq j. \tag{A.1}$$

If the characteristic function of the (n-1)-tuple $(\ln \tilde{x}_{ij},\ldots,\ln \tilde{x}_{(j-1)j}, \ln \tilde{x}_{(j+1)j},\ldots,\ln \tilde{x}_{nj})$ does not vanish, then the distribution of $(\tilde{x}_{1j},\ldots,\tilde{x}_{(j-1)j},\tilde{x}_{(j+1)j},\ldots,\tilde{x}_{nj})$ determines the distributions of $v_1,\ldots,v_n$ up to a change in scale.

Proof. Without loss of generality, assume j = n, then the joint characteristic function of the (n-1)-tuple $\ln \tilde{x}_n = (\ln \tilde{x}_{1n},\ldots,\ln \tilde{x}_{(n-1)n})$ is

$$\phi_{\ln x_n}(t_1,\ldots,t_{(n-1)}) = E[\exp i \sum_{k=1}^{n-1} t_j \ln x_k] \tag{A.2}$$

$$= \left\{ \prod_{j=1}^{n-1} \phi_{\ln v_j}(t_j) \right\} \phi_{\ln v_n}(-(t_1 + \ldots + t_{n-1})). \tag{A.3}$$

The condition of nonvanishing $\phi_{\ln x_n}(t_1,\ldots,t_{n-1})$ is equivalent to the nonvanishing of any of the functions $\phi_{\ln v_j}(\cdot)$, $j = 1,\ldots,n$.

Let $\tilde{w}_1,\ldots,\tilde{w}_n$ be another n-tuple of independent real random variables that satisfy the assumptions of the lemma. Let

$$\tilde{y}_{ij} = \frac{\tilde{w}_i}{\tilde{w}_j}, \qquad i,j = 1,\ldots,n,\ i \neq j. \tag{A.4}$$

Again assume that $j = n$. Then the joint characteristic function of the $(n-1)$-tuple $\ln \tilde{y}_n = (\ln \tilde{y}_{1n},\ldots,\ln \tilde{y}_{(n-1)n})$ is

$$\psi_{\ln y_n}(t_1,\ldots,t_{n-1}) = E[\exp i \sum_{l=1}^{n-1} t_l \ln y_l] \tag{A.5}$$

$$= \psi_{\ln w_n}(-(t_1 + \ldots + t_{n-1})) \prod_{l=1}^{n-1} \psi_{\ln w_l}(t_l). \tag{A.6}$$

Assume that both the $(n-1)$-tuples $(\ln \tilde{v}_1,\ldots,\ln \tilde{v}_{n-1})$ and $(\ln \tilde{w}_1,\ldots,\ln \tilde{w}_{n-1})$ have the same j.p.d.f., so that they share the same joint characteristic function. Thus,

$$\phi_{\ln v_n}(-(t_1 + \ldots + t_{n-1})) \prod_{j=1}^{n-1} \phi_{\ln v_j}(t_j)$$

$$= \psi_{\ln w_n}(-(t_1 + \ldots + t_{n-1})) \prod_{j=1}^{n-1} \psi_{\ln w_j}(t_j). \tag{A.7}$$

Assume that

$$\psi_{\ln w_j}(t_j) = \phi_{\ln v_j}(t_j)\, \zeta_j(t_j) \qquad j = 1,\ldots,n-1. \tag{A.8}$$

Substituting (A.8) into (A.7) yields the following

$$\zeta_n(-(t_1 + \ldots + t_{n-1})) \prod_{j=1}^{n-1} \zeta_j(t_j) = 1 \qquad (A.9)$$

in which $\zeta_j(t_j)$ are unknown complex functions continuous on the whole line $-\infty < t_j < \infty$, $j = 1,\ldots,n-1$, satisfying

$$\zeta_j(0) = 1 \qquad j = 1,\ldots,n-1. \qquad (A.10)$$

Equation (A.9) can be solved by putting $t_i = t$, $t_j = 0$, $i = 1,\ldots,n-1$, $j \neq i$, which after using (A.10) yields

$$\zeta_i(t)\zeta_n(-t) = 1 \qquad i = 1,\ldots,n-1. \qquad (A.11)$$

Putting $\zeta_i(t)$, $i = 1,\ldots,n-1$, given in (A.11) into equation (A.9), and changing signs yields

$$\zeta_n(t_1 + \ldots + t_n) = \prod_{j=1}^{n-1} \zeta_n(t_j). \qquad (A.12)$$

The only continuous function satisfying (A.12) and (A.10) is the exponential function

$$\zeta_n(t) = e^{st} \qquad -\infty < t < \infty \qquad (A.13)$$

where s is a complex number, i.e., $s = a + ib$. Putting (A.13) into equation (A.11) yields

$$\zeta_j(t) = e^{st} \qquad j = 1,\ldots,n. \qquad (A.14)$$

Putting (A.14) into (A.8) yields

$$\psi_{\ln w_j}(t_j) = e^{st}\, \phi_{\ln v_j}(t_j) \qquad j = 1,\ldots,n-1. \qquad (A.15)$$

A known property of characteristic function is that

$$\psi_z(t) = \overline{\psi}_z(t) \tag{A.16}$$

where $\overline{\psi}_z(t)$ is the complex conjugate of $\psi_z(t)$. Thus, is follows that the complex number s in (A.15) must be of the form

$$s = ib \tag{A.17}$$

where b is a real constant. Hence

$$\psi_{\ln w_j}(t_j) = e^{ib}\, \phi_{\ln v_j}(t_j). \tag{A.18}$$

Equation (A.18) can be interpreted to mean that the distributions of v_j and w_j are the same up to a change of location, j = 1,...,n-1.

□

Proof of Theorem (Kotlarski, 1967)

Let $\tilde{v}_1,\ldots,\tilde{v}_n$ be independent gamma variates with p.d.f.

$$f_i(v_i) = \begin{cases} \dfrac{v_i^{\alpha_i - 1}\, e^{-v_i/\beta}}{\beta^{\alpha_i}\Gamma(\alpha_i)}, & v_j > 0 \\ 0, & \text{elsewhere} \end{cases} \tag{A.19}$$

where β, $\alpha_i > 0$, i = 1,...,n. Also let

$$\tilde{x}_{ij} = \tilde{v}_i/\tilde{v}_j \qquad i,j = 1,\ldots,n,\ i \neq j \tag{A.20}$$

and

$$\tilde{z}_k = \ln \tilde{v}_k \qquad k = 1,\ldots,n. \tag{A.21}$$

The characteristic function of the random variable $\tilde{z}_k$ is given by

$$\phi_{z_k}(t_k) = E[\exp\, it_k z_k] = \beta^{it} \frac{\Gamma(\alpha_k + it)}{\Gamma(\alpha_k)} \qquad k = 1,\ldots,n. \qquad (A.22)$$

Without loss of generality let $j = n$, then the joint characteristic function of the $(n-1)$-tuple $(\ln \tilde{x}_{1n},\ldots,\ln \tilde{x}_{(n-1)n})$ is given by

$$\phi_{\ln x_n}(t_1,\ldots,t_{n-1}) = E[\exp\, i \sum_{k=1}^{n-1} t_k \ln x_{kn}]$$

$$= \left[\frac{\Gamma(\alpha_n - i \sum_{k=1}^{n-1} t_k)}{\Gamma(\alpha_n)}\right] \prod_{k=1}^{n-1} \frac{\Gamma(\alpha_k + it_k)}{\Gamma(\alpha_k)}. \qquad (A.23)$$

The joint characteristic function of the $(n-1)$-tuple $(\ln \tilde{x}_{1n},\ldots,$ $\ln \tilde{x}_{(n-1)n})$, assuming that the vector random variable $(\tilde{x}_{1n},\ldots,\tilde{x}_{(n-1)n})$ is distributed as an $(n-1)$-variate inverted Dirichlet distribution $D'(\alpha_1,\ldots,\alpha_{n-1};\ \alpha_n)$, is identical with that given in (A.23). This result, with the Lemma, proves the theorem.

□

APPENDIX B
MATHEMATICAL APPENDIX TO CHAPTER IV

The appendix contains the proofs of a corollary and a lemma used in the development of the probability

$$P\left(\bigcap_{\substack{j=1\\ j\neq 1}}^{n} E_i > E_j \,\middle|\, \alpha_1,\ldots,\alpha_n\right)$$

where $E = \{E_1,\ldots,E_n\}$ is the set of decision elements under comparison, and $\alpha_1,\ldots,\alpha_n$ are the parameters characterizing the distributions of the uncertain relative importance judgment ratios $\tilde{x}_{ij}$, $i,j = 1,\ldots,n$, $i \neq j$. Appendix B.1 is concerned with the proof of a corollary which establishes a generalized result of a product of n absolutely convergent series. Appendix B.2 proves an important lemma (Lemma 4.1) which establishes a general computational formula for the calculation of the probability

$$P\left(\bigcap_{\substack{j=1\\ j\neq 1}}^{n} E_i > E_j \,\middle|\, \alpha_1,\ldots,\alpha_n\right)$$

using the j.p.d.f. of the j^{th} column of the pairwise comparison matrix $\tilde{X}$.

Appendix B.1

Theorem B.1.1. If $\sum_{n=0}^{\infty} a_n$ converges absolutely to A, then any rearrangement $\sum_{n=0}^{\infty} b_n$ of $\sum_{n=0}^{\infty} a_n$ also converges absolutely to A (Goldberg, 1976, p. 78).

The theorem of rearrangements leads to a theorem on the multiplication of series.

Theorem B.1.2. If the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absolutely to A and B, respectively, then

$$(\sum_{n=0}^{\infty} a_n)(\sum_{n=0}^{\infty} b_n) = \sum_{n=0}^{\infty} c_n. \tag{B.1.1}$$

The series $\sum_{n=0}^{\infty} c_n$ converges absolutely to AB, where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \qquad n = 0,1,2,... \tag{B.1.2}$$

(Goldberg, 1976, p. 79).

Corollary B.1.1. If for some $x \in R$, the power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ are absolutely convergent, then

$$(\sum_{n=0}^{\infty} a_n x^n)(\sum_{n=0}^{\infty} b_n x^n) = \sum_{n=0}^{\infty} c_n x^n . \tag{B.1.3}$$

The series $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely, where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \qquad n = 1,2,\ldots \tag{B.1.4}$$

(Goldberg, 1976, p. 80).

Corollary B.1.1 can be generalized for the case of a product of n power series.

Corollary B.1.2. If for some $x \in R$, the power series

$$\sum_{k_1=0}^{\infty} (a_1)_{k_1} x^{k_1}, \ldots, \sum_{k_{n-1}=0}^{\infty} (a_{n-1})_{k_{n-1}} x^{k_{n-1}}, \text{ and } \sum_{k_n=0}^{\infty} (a_n)_{k_n} x^{k_n}$$

are absolutely convergent, then

$$\prod_{i=1}^{n} \sum_{k_i=0}^{\infty} (a_i)_{k_i} x^{k_i} =$$

$$\sum_{k_n=0}^{\infty} \sum_{k_{n-1}=0}^{k_n} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} (a_1)_{k_1} (a_2)_{k_2-k_1} \cdots (a_{n-1})_{k_{n-1}-k_{n-2}}$$

$$(a_n)_{k_n-k_{n-1}} x^{i_n} . \tag{B.1.5}$$

Proof (by induction)

Suppose that the following product relationship is true for n-1

$$\prod_{i=1}^{n-1} \sum_{k_i=0}^{\infty} (a_i)_{k_i} x^{k_i} =$$

$$\sum_{k_{n-1}=0}^{\infty} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_1=0}^{k_2} (a_1)_{k_1} (a_2)_{k_2-k_1} \cdots (a_{n-1})_{k_{n-1}-k_{n-2}} x^{k_{n-1}} . \tag{B.1.6}$$

For notational convenience, define

$$(b_{n-1})_{k_{n-1}} = \sum_{k_{n-2}=0}^{k_{n-1}} \sum_{k_{n-3}=0}^{k_{n-2}} \cdots \sum_{k_1=0}^{k_2} (a_1)_{k_1} (a_2)_{k_2-k_1} \cdots (a_{n-1})_{k_{n-1}-k_{n-2}} \tag{B.1.7}$$

for $k_{n-1} = 0,1,\ldots$.

Then equation (B.1.6) can be rewritten as

$$\prod_{i=1}^{n-1} \sum_{k_i=0}^{\infty} (a_i)_{k_i} x^{k_i} = \sum_{k_n=0}^{\infty} (b_{n-1})_{k_{n-1}} x^{k_{n-1}}. \tag{B.1.8}$$

Using (B.1.8) equation (B.1.5) can be written as

$$\prod_{i=1}^{n} \sum_{k_i=0}^{\infty} (a_i)_{k_i} x^{k_i} = \left(\sum_{k_n=0}^{\infty} (a_n)_{k_n} x^{k_n} \right) \left(\sum_{k_{n-1}=0}^{\infty} (b_{n-1})_{k_{n-1}} x^{k_{n-1}} \right). \tag{B.1.9}$$

This product can be rewritten, using the results of Corollary B.1.1, as

$$\prod_{i=1}^{n} \sum_{k_i=0}^{\infty} (a_i)_{k_i} x^{k_i} = \sum_{k_n=0}^{\infty} \sum_{k_{n-1}=0}^{k_n} (b_{n-1})_{k_{n-1}} (a_n)_{k_n-k_{n-1}} x^{k_n}. \tag{B.1.10}$$

Substituting equation (B.1.7) for $(b_{n-1})_{k_{n-1}}$ yields

$$\prod_{i=1}^{n} \sum_{k_i=0}^{\infty} (a_i)_{k_i} x^{k_i} =$$

$$\sum_{k_n=0}^{\infty} \sum_{k_{n-1}=0}^{k_n} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_1=0}^{k_2} (a_1)_{k_1} (a_2)_{k_2-k_1} \cdots (a_n)_{k_n-k_{n-1}} x^{k_n}. \tag{B.1.11}$$

Hence, if (B.1.5) is true for n-1, it is also true for n, and since it is true for n = 2, it must be true for all n by induction.

□

Corollary B.1.2 is useful for extending the product of (n-1) absolutely convergent power series found in equation (4.9); that is,

$$\prod_{j=2}^{n}\sum_{k_j=0}^{\infty}\frac{e^{-v_i/\beta}(v_1/\beta)^{k_j+\alpha_j}}{(\alpha_j+k_j)!} = e^{-(n-1)v_1/\beta}\left(\frac{v_1}{\beta}\right)^{\alpha_2+\ldots+\alpha_n}\prod_{j=2}^{n}\sum_{k_j=0}^{\infty}\frac{(v_i/\beta)^{k_j}}{(\alpha_j+k_j)!}. \tag{B.1.12}$$

Using the result found in Corollary B.1.2, with

$$\begin{aligned}(a_2)_{k_2} &= (\alpha_2+k_2)^{-1}\\ (a_3)_{k_3-k_2} &= (\alpha_3+k_3-k_2)\\ &\vdots\\ (a_n)_{k_n-k_{n-1}} &= (\alpha_n+k_n-k_{n-1})^{-1}\end{aligned} \tag{B.1.13}$$

equation (B.1.12) can be written as

$$\prod_{j=2}^{n}\sum_{k_j=0}^{\infty}\frac{e^{-v_1/\beta}(v_1/\beta)^{k_j+\alpha_j}}{(\alpha_j+k_j)!} =$$

$$\sum_{k_n=0}^{\infty}\sum_{k_{n-1}=0}^{k_n}\cdots\sum_{k_2=0}^{k_3}\frac{e^{-(n-1)v_1/\beta}(v_1/\beta)^{\alpha_2+\ldots+\alpha_n+k_n}}{(\alpha_2+k_2)!(\alpha_3+k_3+k_2-1)!\ldots(\alpha_n+k_n-k_{n-1})!}.$$

Appendix B.2

Lemma 4.1. Let α_i, $i = 1,\ldots,n-1$, be positive integers, then

$$\begin{aligned}P_n(1;\alpha_1,\ldots,\alpha_n;\alpha_{n+1}) &= P_1(1;\alpha_1;\alpha_{n+1}) - \sum_{n+1>i_{k_2}>1} S_{\alpha_1}(\alpha_{i_{k_2}})\\ &+ \sum_{n+1>i_{k_3}>i_{k_2}>1} S_{\alpha_1}(\alpha_{i_{k_3}},\alpha_{i_{k_2}}) - \sum_{n+1>i_{k_4}>i_{k_3}>i_{k_2}>1} S_{\alpha_1}(\alpha_{i_{k_4}},\alpha_{i_{k_3}},\alpha_{i_{k_2}})\\ &+ \ldots + (-1)^{n+1}S_{\alpha_1}(\alpha_n,\ldots,\alpha_2)\end{aligned} \tag{B.2.1}$$

where

$$S_{\alpha_1}(\alpha_{i_{k_2}}) = \sum_{i_{k_2}=0}^{\alpha_{k_2}-1} \binom{\alpha_1+i_{k_2}-1}{\alpha_1-1} P_1(2;\alpha_1+i_{k_2};\alpha_{n+1}), \tag{B.2.2}$$

$$S_{\alpha_1}(\alpha_{i_{k_3}},\alpha_{i_{k_2}}) = \sum_{i_{k_3}=0}^{\alpha_{k_3}-1} \sum_{i_{k_2}=0}^{\alpha_{k_2}-1} \binom{\alpha_1+i_{k_3}+i_{k_2}-1}{\alpha_1-1,i_{k_3},i_{k_2}} P_1(2;\alpha_1+i_{k_3}+i_{k_2};\alpha_{n+1}), \tag{B.2.3}$$

$$\text{for } n+1 > k_3 > k_2 > 1,$$

$$S_{\alpha_1}(\alpha_{i_{k_4}},\alpha_{i_{k_3}},\alpha_{i_{k_2}}) =$$

$$\sum_{i_{k_4}=0}^{\alpha_{k_4}-1} \sum_{i_{k_3}=0}^{\alpha_{k_3}-1} \sum_{i_{k_2}=0}^{\alpha_{k_2}-1} \binom{\alpha_1+i_{k_4}+i_{k_3}+i_{k_2}-1}{\alpha_1-1,i_{k_4},i_{k_3},i_{k_2}} P_1(3;\alpha_1+i_{k_4}+i_{k_3}+i_{k_2};\alpha_{n+1}), \tag{B.2.4}$$

$$\text{for } n+1 > k_4 > k_3 > k_2 > 1,$$

$$S_{\alpha_1}(\alpha_n,\alpha_{n-1},\ldots,\alpha_2) =$$

$$\sum_{i_n=0}^{\alpha_n-1} \sum_{i_{n-1}=0}^{\alpha_{n-1}-1} \cdots \sum_{i_2=0}^{\alpha_2-1} \binom{\alpha_1+i_n+i_{n-1}+\ldots+i_2}{\alpha_1-1,i_n,\ldots,i_2} P_1(n-1;\alpha_1+i_n+\ldots+i_2;\alpha_{n+1}), \tag{B.2.5}$$

and

$$P_1(m;\alpha_1+i_{k_2}+\ldots+i_{k_{n-1}};\alpha_{n+1}) =$$

$$\binom{\alpha_1+i_{k_1}+\ldots+i_{k_{m-1}}}{m} \sum_{i_1=0}^{\alpha_1-1} \binom{\alpha_{n+1}+i_1-1}{\alpha_{n+1}-1} \left(\frac{m}{m+1}\right)^{i_n} \left(\frac{1}{m+1}\right)^{\alpha_{n+1}}. \tag{B.2.6}$$

Proof

Before proving Lemma 4.1, equation (B.2.6) will be established.

$$P_1(m;\alpha_1+i_{k_2}+\ldots+i_{k_{m-1}};\alpha_{n+1})$$

$$= \frac{\Gamma(\alpha_1+i_{k_2}+\ldots+i_{k_{m-1}}+\alpha_{n+1})}{\Gamma(\alpha_1+i_{k_2}+\ldots+i_{k_{m-1}})\Gamma(\alpha_{n+1})} \int_1^{\infty} \frac{z_1^{\alpha_1+i_{k_2}+\ldots+i_{k_{m-1}}-1}}{(1+mz_1)^{\alpha_1+i_{k_2}+\ldots+i_{k_{m-1}}+\alpha_{n+1}}} dz_1 \qquad (B.2.7)$$

$$= \frac{\Gamma(\alpha_1+i_{k_2}+\ldots+i_{k_{m-1}}+\alpha_{n+1})}{\Gamma(\alpha_1+i_{k_2}+\ldots+i_{k_{m-1}})\Gamma(\alpha_{n+1})} \left(\frac{1}{m}\right)^{\alpha_1+i_{k_2}+\ldots+i_{k_{m-1}}}$$

$$\cdot \int_m^{\infty} \frac{y^{\alpha_1+i_{k_2}+\ldots+i_{k_{m-1}}-1}}{(1+y)^{\alpha_1+i_{k_2}+\ldots+i_{k_{m-1}}+\alpha_{n+1}}} dz_1 \qquad (B.2.8)$$

by change of variable, i.e., $y = mz_1$. Equation (B.2.8) can be equivalently written as

$$P_1(m;\alpha_1+i_m;\alpha_{n+1}) =$$

$$\left(\frac{1}{m}\right)^{\alpha_1+i_m} \left\{1 - \frac{\Gamma(\alpha_1+i_m+\alpha_{n+1})}{\Gamma(\alpha_1+i_m)\Gamma(\alpha_{n+1})} \int_0^m \frac{y^{\alpha_1+i_m-1}}{(1+y)^{\alpha_1+i_m+\alpha_{n+1}}} dy\right\}, \qquad (B.2.9)$$

where $i_m = i_{k_1}+\ldots+i_{k_{m-1}}$. Evaluating (B.2.9) using (4.20), yields

$$P_1(m;\alpha_1+i_m;\alpha_{n+1}) =$$

$$\left(\frac{1}{m}\right)^{\alpha_1+i_m} \frac{\Gamma(\alpha_1+i_m+\alpha_{n+1})}{\Gamma(\alpha_1+i_m)\Gamma(\alpha_{n+1})} \sum_{i_1=0}^{\alpha_1+i_n-1} \binom{\alpha_1+i_m-1}{i_1} m^{i_1} \left(\frac{1}{m+1}\right)^{\alpha_{n+1}+i_1}$$

$$\cdot B(\alpha_{n+1}+i_1, \alpha_1+i_m-i_1). \qquad (B.2.10)$$

After expansion of terms and cancellation, equation (B.2.10) becomes

$$P_1(m;\alpha_1+i_{k_2}+\ldots+i_{k_{m-1}};\alpha_{n+1}) =$$

$$\left(\frac{1}{m}\right)^{\alpha_1+i_{k_2}+\ldots+i_{k_{m-1}}} \sum_{i_1=0}^{\alpha_1+i_{k_2}+\ldots+i_{k_{m-1}}-1} \binom{\alpha_{n+1}+i_1-1}{\alpha_{n+1}-1} \left(\frac{m}{m+1}\right)^{i_1} \left(\frac{1}{m+1}\right)^{\alpha_{n+1}}. \tag{B.2.11}$$

The lemma will now be proved by induction. Rewriting (4.29) we have

$$P_n(1;\alpha_1,\ldots,\alpha_n;\alpha_{n+1}) = P_{n-1}(1;\alpha_1,\ldots,\alpha_{n-1};\alpha_{n+1})$$

$$- \sum_{i_n=0}^{\alpha_n-1} \binom{\alpha_n+i_n-1}{\alpha_n-1} P_{n-1}(2;\alpha_1+i_n,\alpha_2,\ldots,\alpha_{n-1};\alpha_{n+1}). \tag{B.2.12}$$

Now, assume that the following equation is true for n-1

$$P_{n-1}(1;\alpha_1,\ldots,\alpha_{n-1};\alpha_{n+1}) = P_1(1;\alpha_1;\alpha_{n+1}) - \sum_{n>i_{k_2}>1} S_{\alpha_1}(\alpha_{i_{k_2}})$$

$$+ \sum_{n>i_{k_3}>i_{k_2}>1} S_{\alpha_1}(\alpha_{i_{k_3}},\alpha_{i_{k_2}}) + \ldots + (-1)^n S(\alpha_{n-1},\ldots,\alpha_2). \tag{B.2.13}$$

Substituting equation (B.2.13) into equation (B.2.12) yields the following equation:

$$P_n(1;\alpha_1,\ldots,\alpha_n;\alpha_{n+1}) = P_{n-1}(1;\alpha_1,\ldots,\alpha_{n-1};\alpha_{n+1})$$

$$- \sum_{i_n=0}^{\alpha_n-1} \binom{\alpha_1+i_n-1}{\alpha_1-1} P_1(2;\alpha_1+i_n;\alpha_{n+1}) + \sum_{i_n=0}^{\alpha_n-1} \binom{\alpha_1+i_n-1}{\alpha_1-1} \sum_{n>i_{k_2}>1} S_{\alpha_1+i_n}(\alpha_{i_{k_2}})$$

$$- \sum_{i_{n-1}=0}^{\alpha_n-1} \binom{\alpha_1+i_n-1}{\alpha_1-1} \sum_{n>i_{k_3}>i_{k_2}>1} S_{\alpha_1+i_n}(\alpha_{i_{k_3}},\alpha_{i_{k_2}})$$

$$+ \ldots - (-1)^n \sum_{i_n=0}^{\alpha_n-1} \binom{\alpha_1+i_n-1}{\alpha_1-1} S_{\alpha_1+i_n}(\alpha_{n-1},\ldots,\alpha_1). \qquad \text{(B.2.14)}$$

The terms of equation (B.2.14) involving the summations must now be evaluated. Thus,

$$\sum_{i_n=0}^{\alpha_n-1} \binom{\alpha_1+i_n-1}{\alpha_1-1} P_1(2;\alpha_1+i_n;\alpha_{n+1}) = S_{\alpha_1}(\alpha_n) \quad \text{(via equation (B.2.2))}, \qquad \text{(B.2.15)}$$

$$\sum_{i_n=0}^{\alpha_n-1} \binom{\alpha_1+i_n-1}{\alpha_1-1} \sum_{n>i_{k_2}>1} S_{\alpha_1+i_n}(\alpha_{i_{k_2}}) =$$

$$\sum_{i_n=0}^{\alpha_n-1} \binom{\alpha_1+i_n-1}{\alpha_1-1} \sum_{n>k_2>1} \sum_{i_{k_2}=0}^{\alpha_{k_2}-1} \binom{\alpha_1+i_n+i_{k_2}-1}{\alpha_1-1,i_n,i_{k_2}}$$

$$\cdot P_1(1;\alpha_1+i_n+i_{k_2};\alpha_{n+1}). \qquad \text{(B.2.16)}$$

Upon rearranging equation (B.2.16) becomes

$$\sum_{i_n=0}^{\alpha_n-1} \binom{\alpha_1+i_n-1}{\alpha_1-1} \sum_{n>i_{k_2}>1} S_{\alpha_1+i_n}(\alpha_{i_{k_2}}) =$$

$$\sum_{n>k_2>1} \sum_{i_n=0}^{\alpha_n-1} \sum_{i_{k_2}=0}^{\alpha_{k_2}-1} \binom{\alpha_1+i_n+i_{k_2}-1}{\alpha_1-1,i_n,i_{k_2}} P_1(2;\alpha_1+i_n+i_{k_2};\alpha_{n+1}) \qquad \text{(B.2.17)}$$

and from (B.2.3), equation (B.2.17) can be written as

$$\sum_{i_n=0}^{\alpha_n-1} \binom{\alpha_1+i_n-1}{\alpha_1-1} \sum_{n>i_{k_2}>1} S_{\alpha_1+i_n}(\alpha_{i_{k_2}}) = \sum_{n>i_{k_2}>1} S(\alpha_n,\alpha_{i_{k_2}}). \quad \text{(B.2.18)}$$

Using similar arguments for the remaining terms of equation (B.2.14) we have

$$P_n(1;\alpha_1,\ldots,\alpha_n;\alpha_{n+1}) = P_{n-1}(1;\alpha_1,\ldots,\alpha_{n-1};\alpha_{n+1})$$

$$- S_{\alpha_1}(\alpha_n) + \sum_{n>i_{k_2}>1} S_{\alpha_1}(\alpha_n,\alpha_{i_{k_2}})$$

$$+ \sum_{n>i_{k_3}>i_{k_2}>1} S_{\alpha_1}(\alpha_n,\alpha_{i_{k_3}},\alpha_{i_{k_2}}) + \ldots + (-1)^{n+1} S_{\alpha_1}(\alpha_n,\ldots,\alpha_2). \quad \text{(B.2.19)}$$

Putting (B.2.13) into (B.2.19), and rearranging, yields

$$P_n(1;\alpha_1,\ldots,\alpha_n;\alpha_{n+1}) = P_1(1;\alpha_1;\alpha_{n+1}) - \sum_{n>i_{k_2}>1} S_{\alpha_1}(\alpha_{i_{k_2}}) - S_{\alpha_1}(\alpha_n)$$

$$+ \sum_{n>i_{k_3}>i_{k_2}} S_{\alpha_1}(\alpha_{i_{k_3}},\alpha_{i_{k_2}}) + \sum_{n>i_{k_2}>1} S_{\alpha_1}(\alpha_n,\alpha_{i_{k_2}})$$

$$- \sum_{n>i_{k_4}>i_{k_3}>i_{k_2}} S_{\alpha_1}(\alpha_{i_{k_4}},\alpha_{i_{k_3}},\alpha_{i_{k_2}})$$

$$- \sum_{n>i_{k_3}>i_{k_2}} S_{\alpha_1}(\alpha_n,\alpha_{i_{k_3}},\alpha_{i_{k_2}})$$

$$+ \ldots + (-1)^{n+1} S_{\alpha_1}(\alpha_n,\ldots,\alpha_2). \quad \text{(B.2.20)}$$

Hence equation (B.2.20) can be written as

$$P_n(1;\alpha_1,\ldots,\alpha_n;\alpha_{n+1}) = P_1(1;\alpha_1;\alpha_{n+1}) - \sum_{n+1>i_{k_2}>1} S_{\alpha_1}(\alpha_{i_{k_2}})$$

$$+ \sum_{n+1>i_{k_3}>i_{k_2}>1} S_{\alpha_1}(\alpha_{i_{k_3}},\alpha_{i_{k_2}}) + \ldots + (-1)^{n+1} S_{\alpha_1}(\alpha_n,\ldots,\alpha_2). \tag{B.2.21}$$

Therefore, if (B.2.13) is true for n-1, it must follow that (B.2.13) is true for n, but since (B.2.13) is true for n = 1, it is true for any n, by mathematical induction.

□

APPENDIX C
MATHEMATICAL APPENDIX TO CHAPTER V

This appendix summarizes the work of Tiao and Guttman (1965, pp. 793-94) concerning the transformation of the Dirichlet distribution $D(\alpha_1,\ldots,\alpha_{n-1};\ \alpha_n)$ to its corresponding inverted multivariate analog, the inverted Dirichlet distribution $D'(\alpha_1,\ldots,\alpha_{n-1};\ \alpha_n)$.

Assume the vector random variable $\tilde{y} = (\tilde{y}_1,\ldots,\tilde{y}_{n-1}) \sim D(\alpha_1,\ldots,\alpha_{n-1};\ \alpha_n)$ with density

$$g_y(y_1,\ldots,y_{n-1}) = \frac{\Gamma\left(\sum_{i=1}^{n} \alpha_i\right)}{\prod_{i=1}^{n} \Gamma(\alpha_i)} \prod_{j=1}^{n-1} y_j^{\alpha_j - 1} \left(1 - \sum_{i=1}^{n-1} y_i\right)^{\alpha_n - 1} \qquad (C.1)$$

at any point in the simplex: S_{n-1}: $\{(y_1,\ldots,y_{n-1}): y_i > 0$ for $i = 1,\ldots,n-1$, and $\sum_{i=1}^{n-1} y_i \leq 1\}$, in R_{n-1} and zero elsewhere.

Consider the transformation

$$T:\ z_i = \frac{\tilde{y}_i}{\tilde{z}_n},\ i = 1,\ldots,n-1; \qquad \tilde{z}_n = 1 - \sum_{i=1}^{n-1} \tilde{y}_i\ . \qquad (C.2)$$

The inverse transformation is

$$T^{-1}:\ \tilde{y}_i = \tilde{z}_i - \tilde{z}_n,\ i = 1,\ldots,n-1; \qquad z_n = \frac{1}{1 + \sum_{i=1}^{n-1} \tilde{z}_i}\ . \qquad (C.3)$$

The Jacobian J of the transformation (C.2) is

$$|J| = \left|\frac{\partial(\tilde{z}_1,\ldots,\tilde{z}_{n-1})}{\partial(\tilde{y}_1,\ldots,\tilde{y}_{n-1})}\right| = \left(\frac{1}{1+\sum_{i=1}^{n-1}\tilde{z}_i}\right)^n . \qquad (C.4)$$

Upon applying the transformation we have that

$$g_z(z_1,\ldots,z_n) = \frac{\Gamma(\sum_{i=1}^{n}\alpha_i)}{\prod_{i=1}^{n}\Gamma(\alpha_i)} \prod_{j=1}^{n-1} z_j^{\alpha_j-1} \left(1+\sum_{i=1}^{n-1} z_j\right)^{-\sum_{i=1}^{n}\alpha_i} \qquad (C.5)$$

for $0 < z_i < \infty$, $i = 1,\ldots,n-1$, in k_{n-1} and zero elsewhere.

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